Riemann solvers: a brief review

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Exact relation between integral averages

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Integration in space and time on control volume $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right] + \Delta t S_i$$ **Exact relation**

$$\left.\begin{aligned} Q_i^n &= \tfrac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Q(x,0)\,dx \\ F_{i+1/2} &= \tfrac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(\tau))\,d\tau \\ S_i &= \tfrac{1}{\Delta t}\tfrac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x,t))\,dx\,dt \end{aligned}\right\}$$ Integral averages

Evolution of integral averages: finite volume method

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Control volume in computational domain $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right] + \Delta t S_i$$ **Update formula**

Integral average at time n $$Q_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Q(x,0)\,dx$$

Numerical flux $$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(\tau))\,d\tau$$

Numerical source $$S_i = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x,t))\,dx\,dt$$

Classical Riemann solvers:

Exact solvers (Godunov, 1959)
Linearized solvers (Godunov, 1960)
Roe (1981)
Osher and Solomon (1982)
HLL (Harten-Lax-van Leer) (1983)
    Davis (1988)
HLLE Einfeldt (1988)
HLLC (Toro et al. 1992, 1994)
    Toro and Chakraborty 1994)
    Batten et al. (1995,1997)
FVS type solvers (Warming-Beam,
    van Leer, Liou-Steffen,..)
Rusanov (1961)
GFORCE (Toro and Titarev, 2006)

Classification

Complete Riemann solver: its structure contains all waves present in the exact solution

Example: the exact solver
Roe
Osher-Solomom
HLLC (for Euler)

Incomplete Riemann solver: its structure contains less waves than present in the exact solver

Example: HLL (for a 3x3 system)
Rusanov
GFORCE

Classification (cont..)

Non-linear Riemann solver:

the exact solver
Osher-Solomom
HLL
HLLC

Linearized Riemann solver:

Godunov's (1960)
Roe

The ideal solver: non-linear and complete

Warning: an incomplete Riemann solver misrepresents discontinuities associated with intermediate waves, resulting in excessive numerical dissipation, specially for ***long time evolution problems***. High-order methods cannot correct this defect of the Riemann solver.

Roe’s Riemann solver

This solver requires the so-called Roe averages. These have been found for anumber of well-kown systems. For a given complex hyperbolic system this may be an impossible task.

This a complete Riemann solver, its structure contains all the characteristic fields of the exact solver. Therefore its numerical dissipation is minimal for intermediate (slow) characteristic fields.

It is a linear solver and thus suffers from the following difficulties:

- Requires an explicit entropy fix (available for some well-known systems)

- Fails near low density flows (negative densities)

- Further reading: chapter 11 of: Toro E F. Riemann solvers and numerical methods for fluid dynamics. Springer, Third Edition, 2009.

Osher’s Riemann solver

This solver is very complex and is only available for standard hyperbolic systems (Euler equations for ideal equation of state; shallow water equations).

It is more expensive than the exact Riemann solver !!

It requires knowledge of “intermediate” states to start the construction of the approximation (two-rarefaction solver is used).

This is a non-linear solver (no entropy fix needed).

This is a complete Riemann solver.

Performs well for slowly-moving shocks
(T W Roberts. The behabivour of flux difference splitting schemes near slowly-moving shock waves. J. Comput. Physics, 141-160, 1990).

Performs well for low-density flows.

- Further reading: chapter 12 of: Toro E F. Riemann solvers and numerical methods for fluid dynamics. Springer, Third Edition, 2009.

Recent solvers for the classical Riemann problem

The MUSTA approach

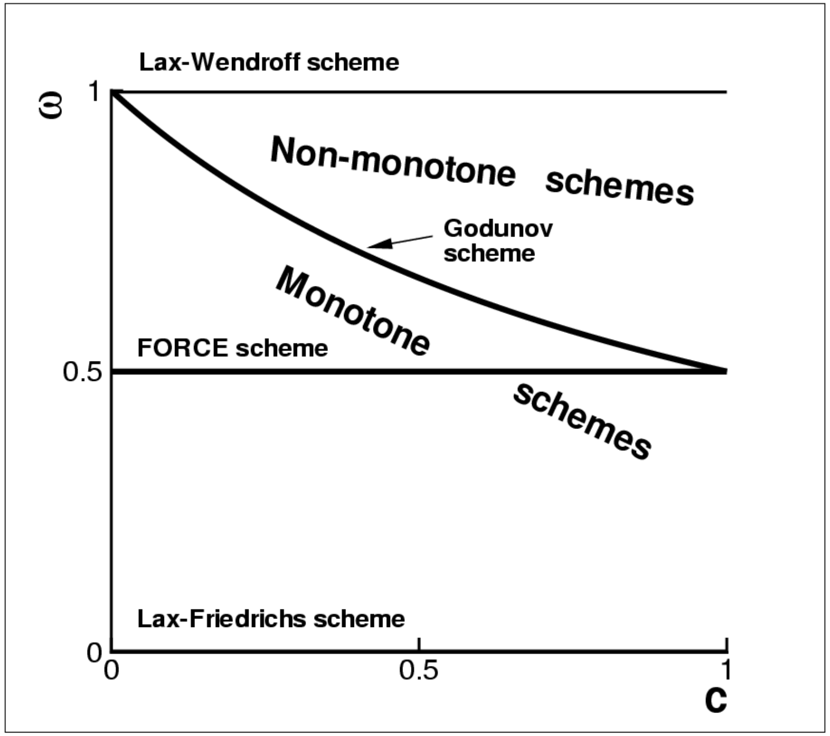
The MUSTA approach (Toro, 2003) is an attempt to regain upwind information but without solving the Riemann problem in the classical sense.

We look for *upwind* schemes that are simple and directly applicable to very complicated problems

A degree of success has been achieved but the work is **not** yet complete, to our satisfaction

Schemes associated to the FORCE flux: a motivating discussion

$$F^{\omega}_{i+1/2} = \omega F^{LW}_{i+1/2} + (1-\omega) F^{LF}_{i+1/2} \,, \quad 0 < \omega < 1$$



Special cases:

$$\omega = 0 \text{ (Lax - Friedrichs)}$$

$$\omega = 1 \text{ (Lax - Wendroff)}$$

$$\omega = 1/2 \text{ (FORCE)}$$

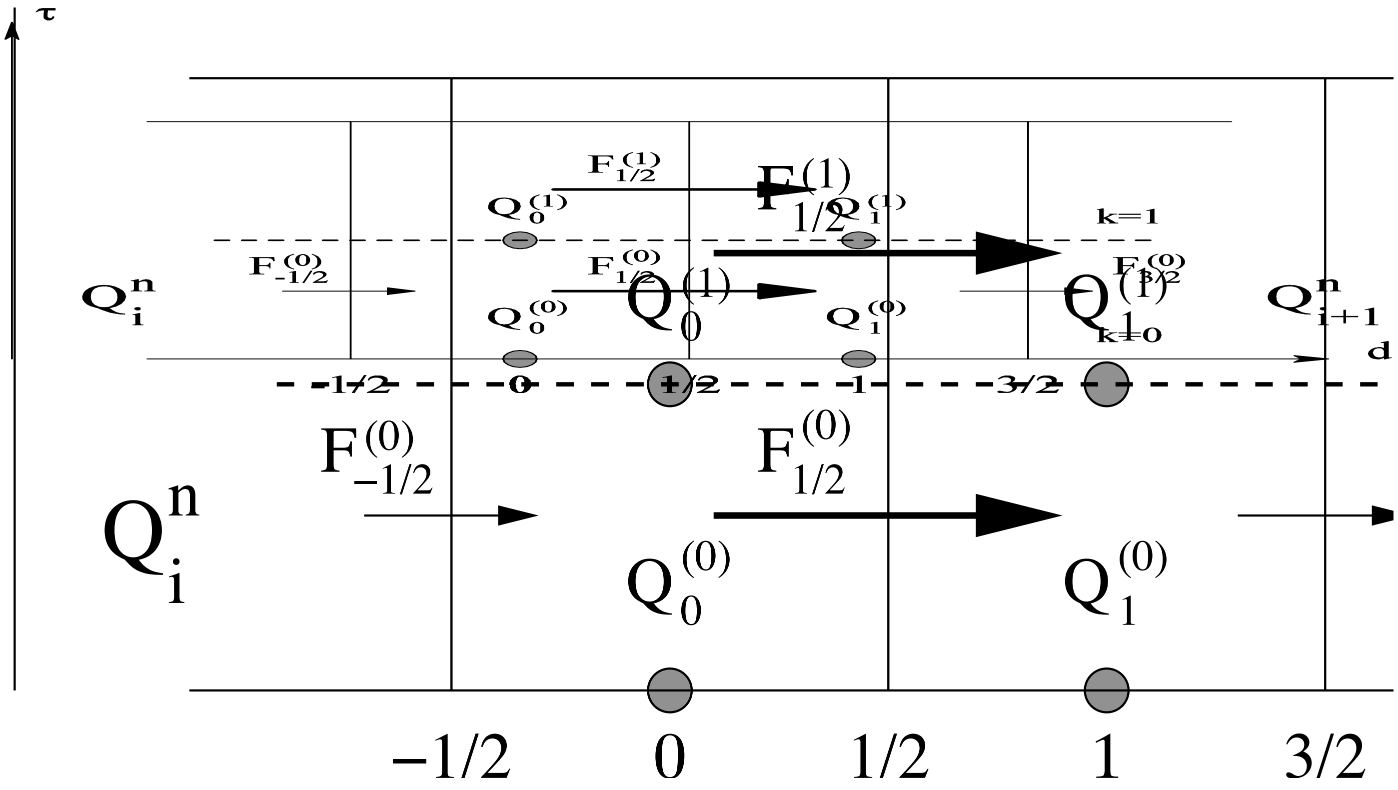
$$\omega = \frac{1}{1+c} \text{ (GFORCE)}$$

Convergence of FORCER scheme in. Cheng and Toro. J Hyp. Diff. Eq. 2004.

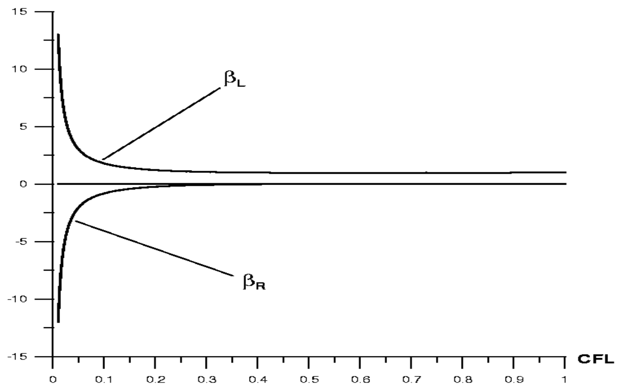
<u>Purpose of MUSTA:</u>

<u>To recover the lost corner by opening the Riemann fan, but without solving the Riemann problem in the classical sense.</u>

MUSTA: multi-stage predictor-corrector approach



Difficulty: Monotonicity!!



This is MUSTA with one stage and 2 local cells using FORCE as predictor and corrector (Toro 2003)

Recent studies (Titarev and Toro, 2006) show that as the number of stages is increased, while keeping the number of cells constant, worsens the situation.

Two classes of MUSTA schemes:

Class 1:

Use FORCE (or GFORCE) as Predictor (many stages on a sufficiently large mesh) and Corrector (final stage) stages

Simple and general but (may be) costly
Research still in progress

More details in:
Toro EF. MUSTA: a multistage numerical flux. Applied Num. Anal. 2006.
Toro E F and Titarev VA. MUSTA schemes for hyperbolic conservation laws. J. Comput. Phys. 2006.
Titarev VA and Toro EF. Int. J. Numer. Meth. Fluids. 2005.

Two classes of MUSTA schemes (cont...)

Class 2: The **EVILIN** variant

- **Perform a predictor stage using FORCE or GFORCE**

- **Perform a linearization on predicted states and solve simple linearized Riemann problem**

  - The resulting Riemann solver is complete
  - It is simple
  - But one needs the eigenstructure of the system
  - It is entropy satisfying

Further details on EVILIN variant in:
Toro E F. Riemann solvers with evolved initial conditions. Int. J. Numer. Meth. In Fluids, Vol. 52, pp 433-453, 2006

The MUSTA-1 scheme

Predictor step

$$\hat{Q}_i^n = Q_i^n - \tfrac{\Delta d}{\Delta\tau}[F^{gforce}(Q_i^n, Q_{i+1}^n) - F(Q_i^n)]$$

$$\hat{Q}_{i+1}^n = Q_{i+1}^n - \tfrac{\Delta d}{\Delta\tau}[F(Q_{i+1}^n) - F^{gforce}(Q_i^n, Q_{i+1}^n)]$$

Corrector step

$$F_{i+1/2}^{GF} = \begin{cases} F(\hat{W}_{1/2}(0)) \\ F^{gforce}(\hat{Q}_i^n, \hat{Q}_{i+1}^n) \end{cases}$$

The EVILIN variant for the 3D Euler equations

Corrector step: solve linear Riemann problem

$$\left.\begin{array}{l} \partial_t W + \hat{B}\partial_x W = 0 \\ W(x,0) = \begin{cases} W_L = W_L(\hat{Q}_i^n) & , x < 0 \\ W_R = W_R(\hat{Q}_{i+1}^n) & , x > 0 \end{cases} \end{array}\right\} \quad \hat{B} = B(\tfrac{1}{2}(\hat{Q}_i^n + \hat{Q}_{i+1}^n))$$

$$\alpha_1 R^{(1)} + \alpha_2 R^{(2)} + \alpha_3 R^{(3)} + \alpha_4 R^{(4)} + \alpha_5 R^{(5)} = \Delta \equiv W_R(\hat{Q}_{i+1}^n) - W_L(\hat{Q}_i^n)$$

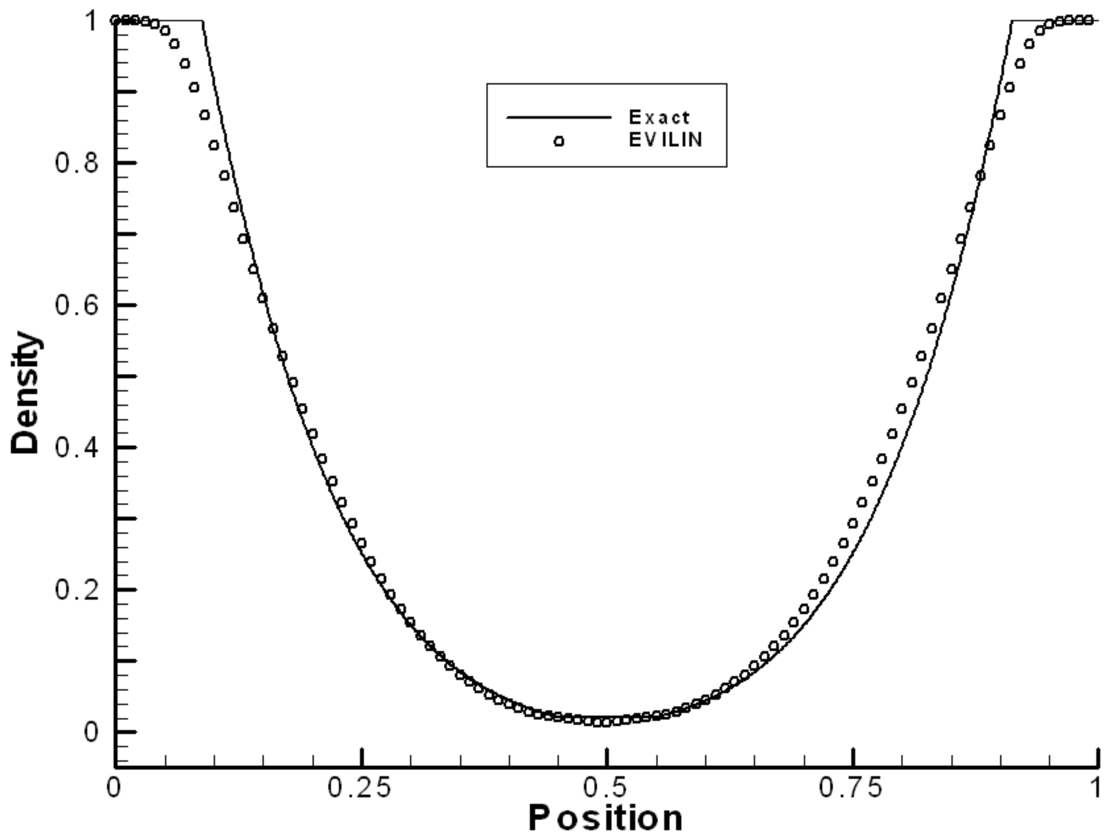
$$\left.\begin{array}{l} \alpha_1 = \dfrac{\Delta p - \Delta u \hat{\rho}\hat{a}}{2\hat{\rho}\hat{a}^2} \\ \alpha_2 = \dfrac{\Delta \rho \hat{a}^2 - \Delta p}{\hat{a}^2} \\ \alpha_3 = \Delta v \\ \alpha_4 = \Delta w \\ \alpha_5 = \dfrac{\Delta p + \Delta u \hat{\rho}\hat{a}}{2\hat{\rho}\hat{a}^2} \\ \Delta q = \hat{q}_R - \hat{q}_L \end{array}\right\}$$

$$\hat{W}_{1/2}(0) = W_L + \sum_{\lambda^{(i)}<0} \gamma_i R^{(i)} \quad \text{or} \quad \hat{W}_{1/2}(0) = W_R - \sum_{\lambda^{(i)}>0} \gamma_i R^{(i)}$$

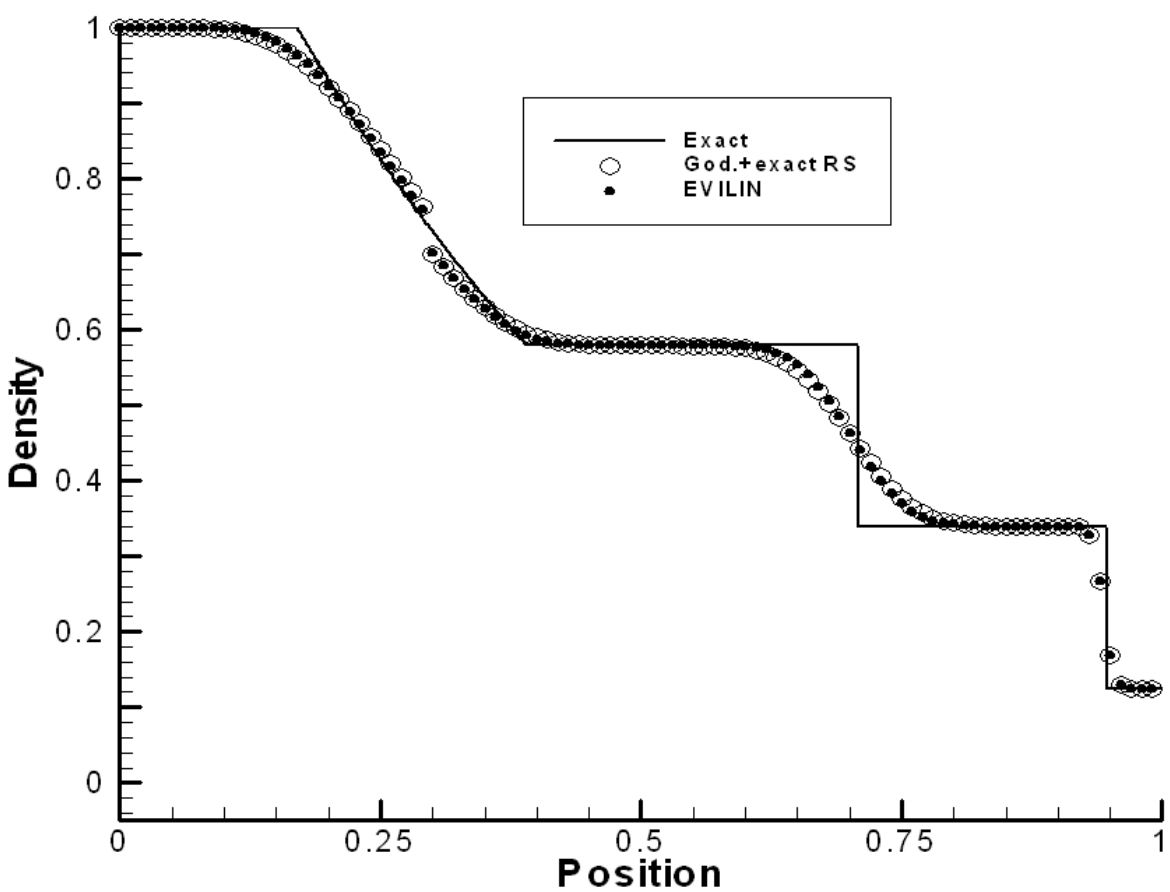
$$\boxed{F_{i+1/2}^{God} = F(\hat{W}_{1/2}(0))}$$

Some test problems for EVILIN applied to the Euler equations.

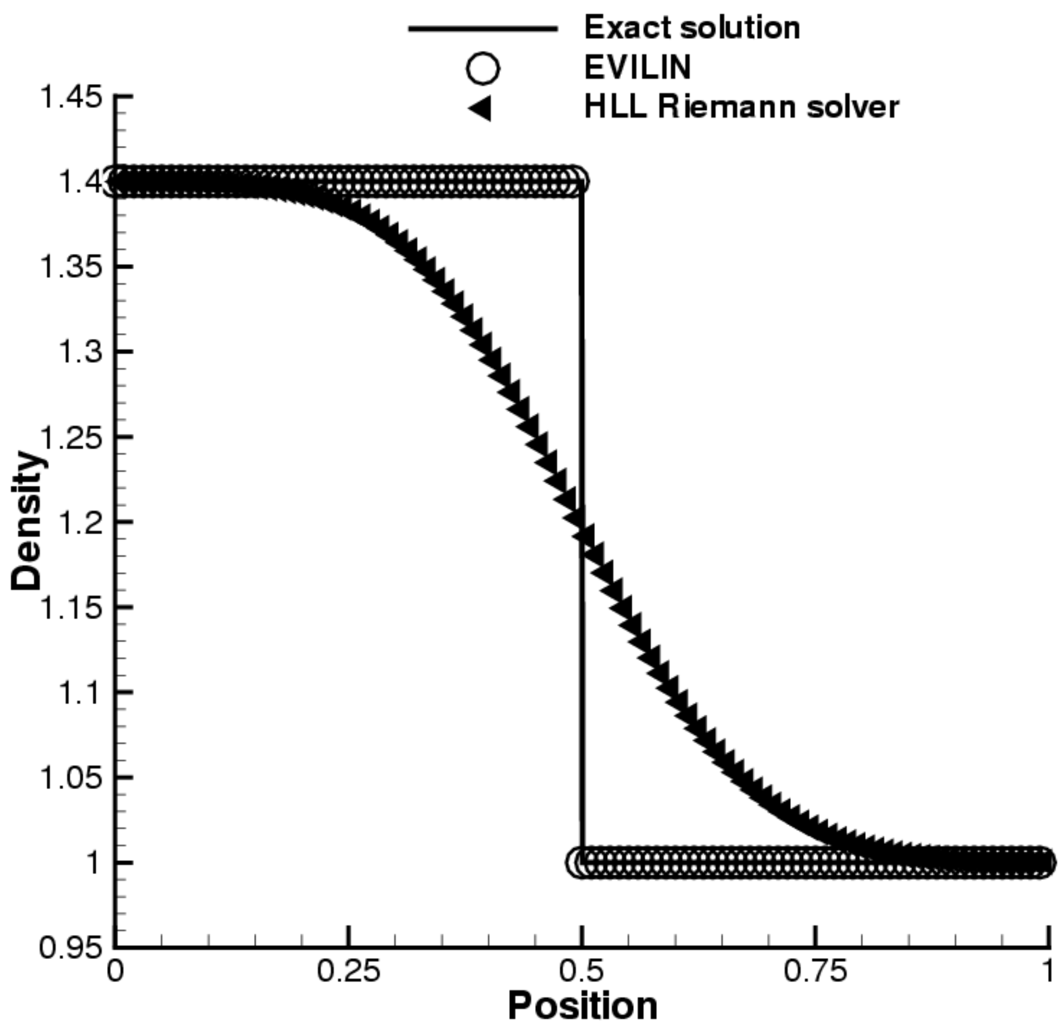
The 123 test (low density)



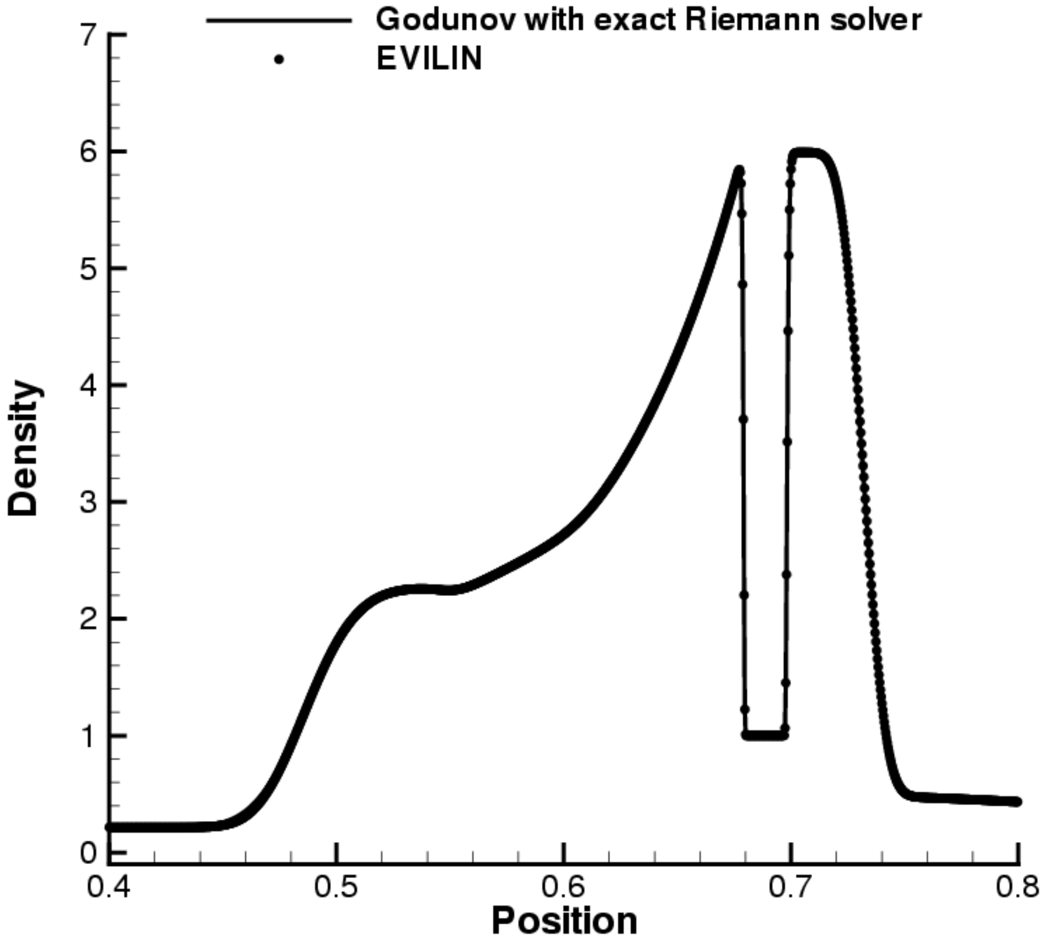
Sonic flow test



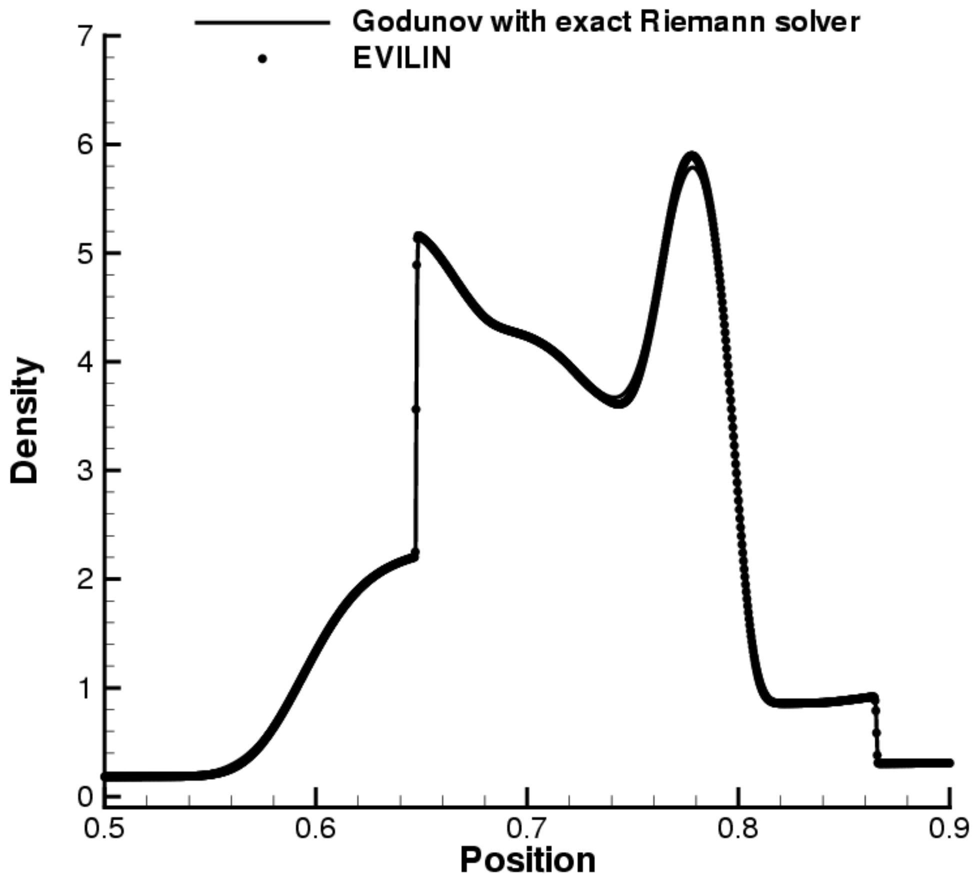
Isolated stationary contact



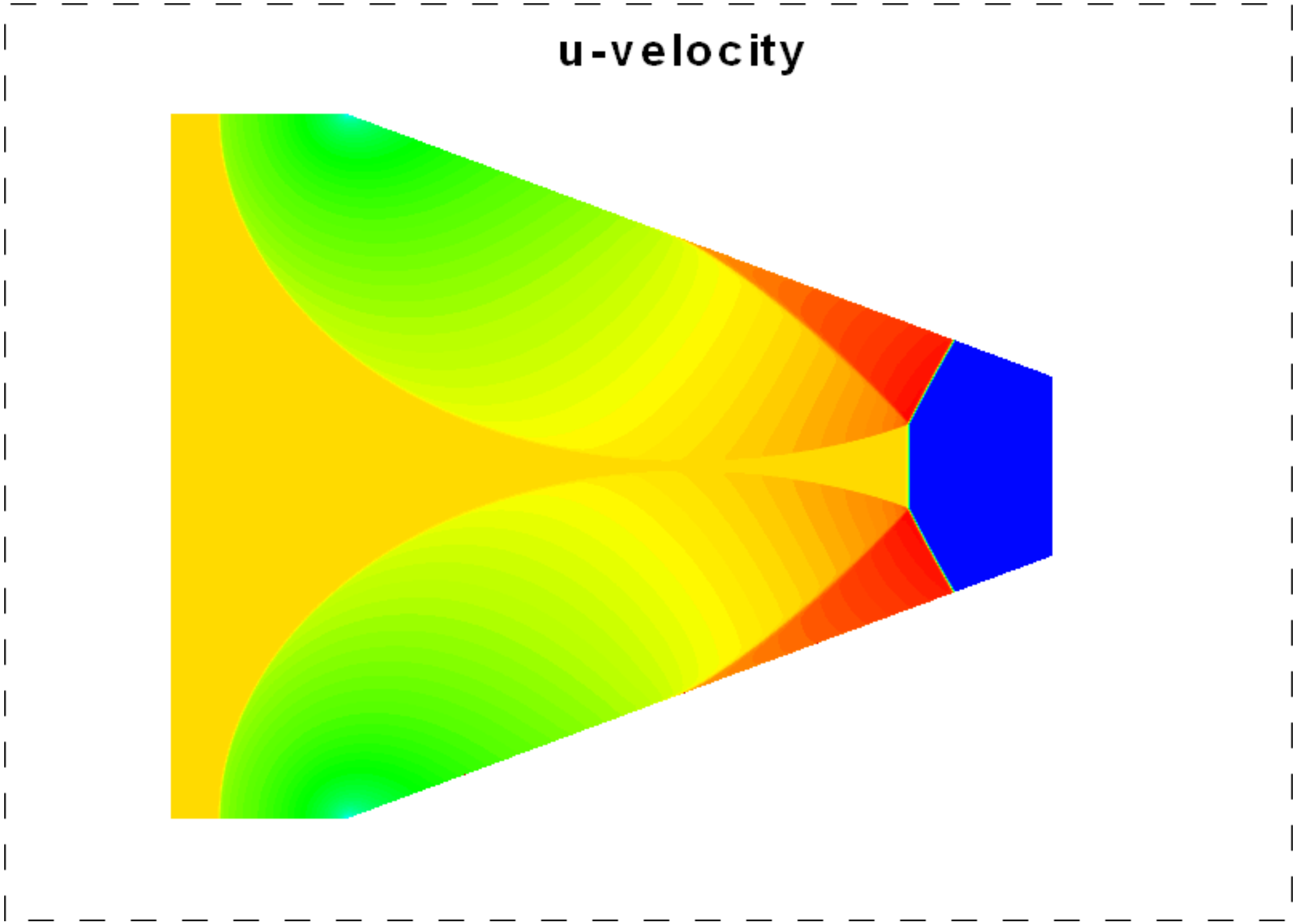
The Woodward and Colella blast wave test



The Woodward and Colella blast wave test



Mach reflection (2D)



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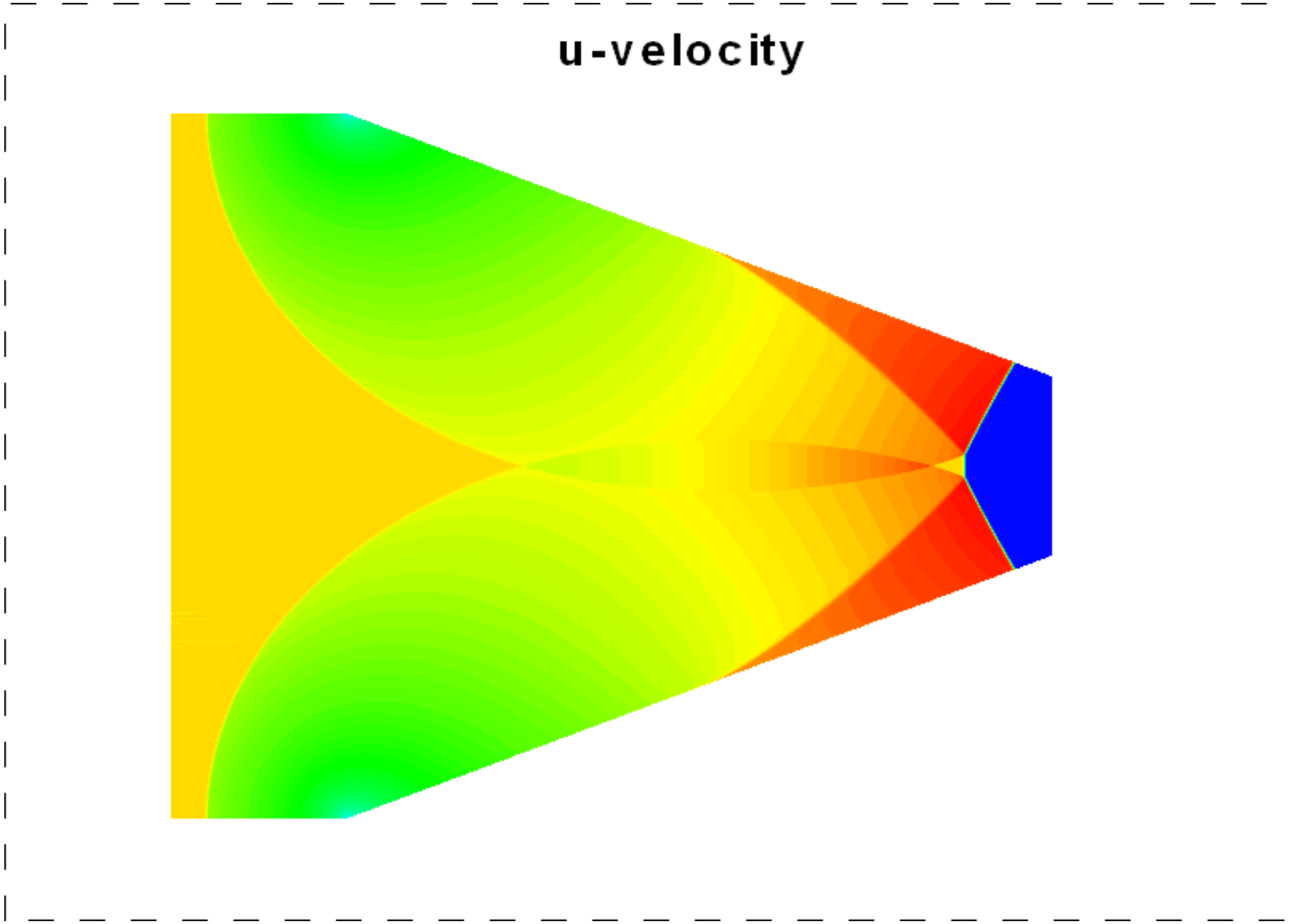

Summary and concluding remarks

- A Riemann solver gives a *numerical flux* (this can be used in finite volume and discontinuous Galerkin finite element methods).
- A Riemann solver (numerical flux) may be *centred* (no explicit use of wave propagation information used) or *upwind* (explicit use of wave propagation information used).
- A Riemann solver may *be linear* or *non-linear*.
- A Riemann solver may be *complete* (all characteristic fields) or *incomplete* (reduced wave model).
- Incomplete Riemann solvers lead to excessive dissipation of slowly moving intermediate waves.
- The ideal solve *is non-lin*ear and *complete*.
- Advances still possible.