ADER high-order schemes for hyperbolic balance laws

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This lecture is about the ADER approach:

(Toro et al. 2001)

**A shock-capturing approach for constructing conservative, non-linear numerical methods of arbitrary accuracy in space and time, on structure and unstructured meshes, in the frameworks of
Finite Volume and
Discontinuous Galerkin Finite Element Methods**

Key feature of ADER:

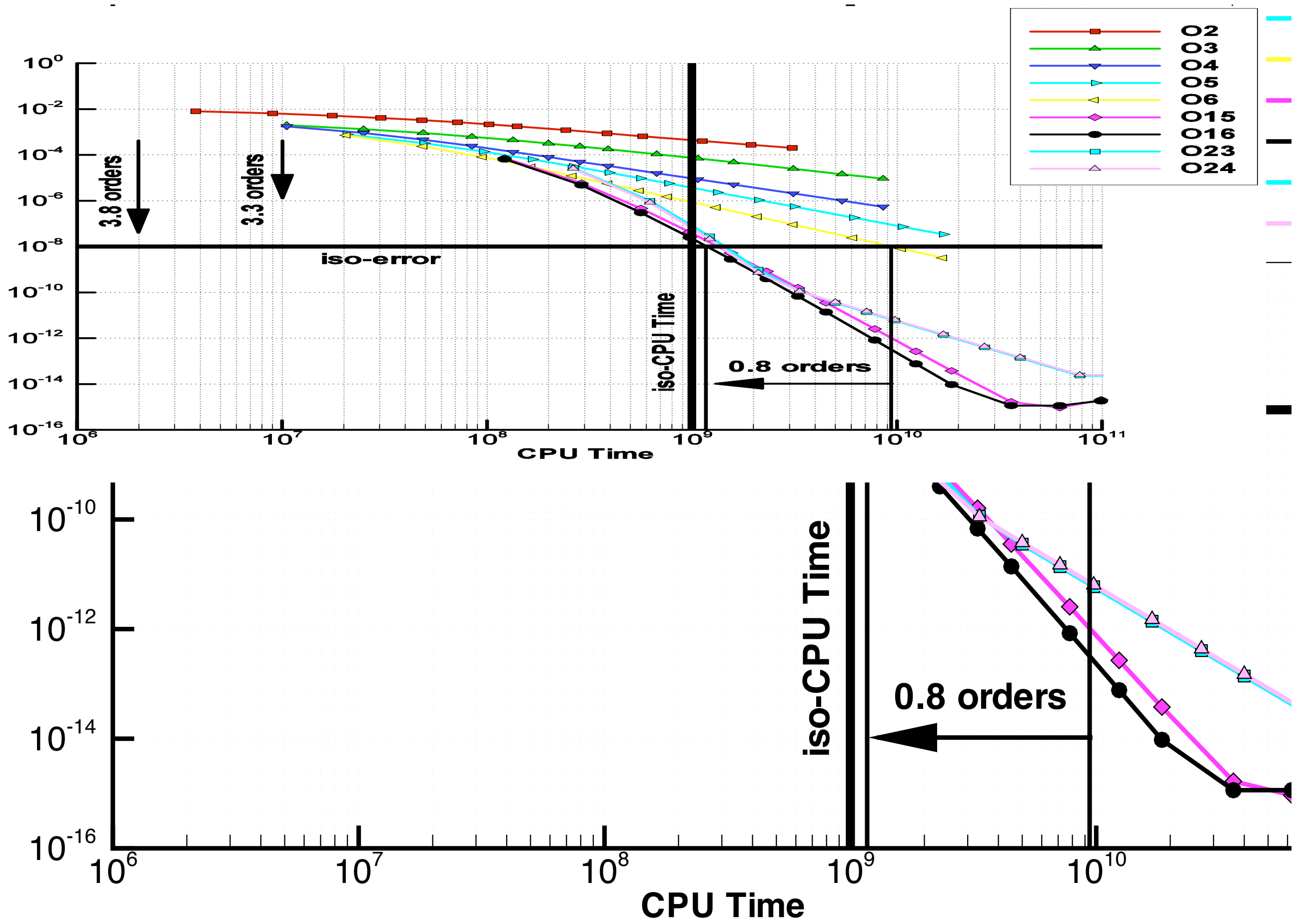
High-order Riemann problem
(also called the *Generalized Riemann problem* or the *Derivative Riemann problem*)

This *generalized Riemann problem* has initial conditions
with a high-order (spatial) representation, such as polynomials

High accuracy.
But why ?

Test for acoustics

ADER



Collaborators: Munz, Schwartzkoppf (Germany), Dumbser (Trento)

Exact relation between integral averages

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Integration in space and time on control volume $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right] + \Delta t S_i$$ **Exact relation**

$$Q_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Q(x,0)\,dx$$

$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(\tau))\,d\tau$$

$$S_i = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x,t))\,dx\,dt$$

Integral averages

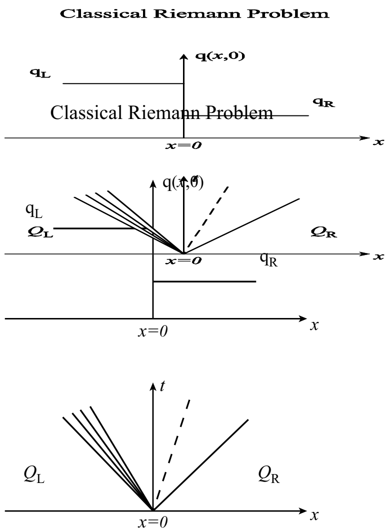
Godunov's finite volume scheme in 1D

(first order accurate)

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right]$$ **Conservative formula**

$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{LR}(\tau))d\tau$$ **Godunov's numerical flux**

$Q_{LR}(\tau)$ **: Solution of classical Riemann problem**



$$\left.\begin{array}{l} \partial_t Q + \partial_x F(Q) = 0 \\ Q(x,0) = \begin{cases} Q_i^n & if \quad x < 0 \\ Q_{i+1}^n & if \quad x > 0 \end{cases} \end{array}\right\} \Rightarrow Q^{(0)}(x/t)$$

$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q^{(0)}(0))d\tau = F(Q^{(0)}(0))$$

Illustration of ADER finite volume method

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Control volume in computational domain $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

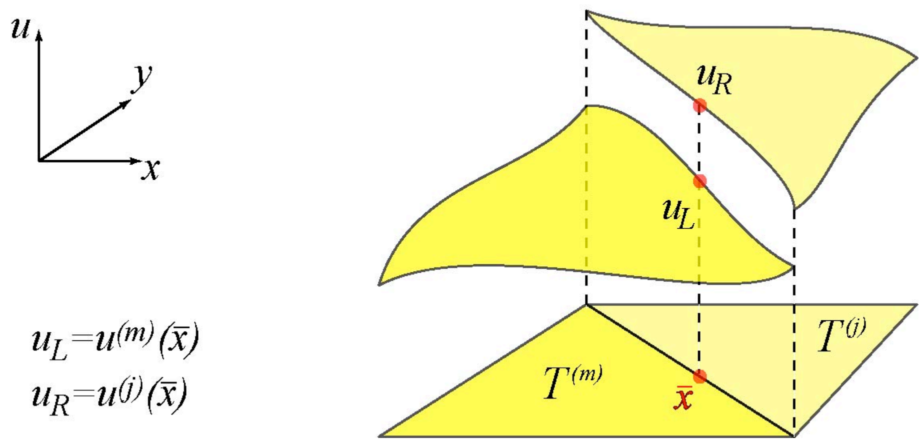
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right] + \Delta t S_i$$ **Update formula**

Integral average at time n
$$Q_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Q(x,0)\,dx$$

Numerical flux
$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(\tau))\,d\tau$$

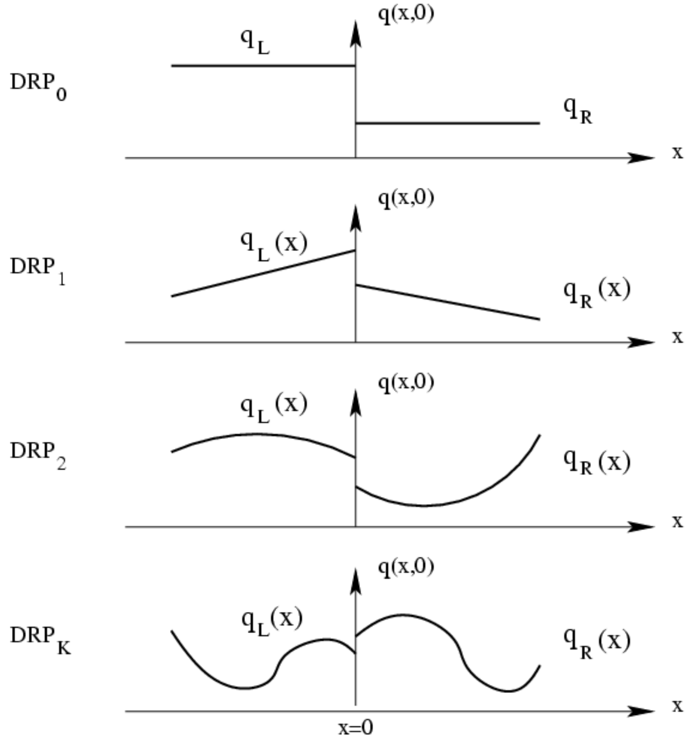
Numerical source
$$S_i = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x,t))\,dx\,dt$$

ADER on 2D unstructured meshes



The numerical flux requires the calculation of an integral in space along
The volume/element interface and in time.

Local Riemann problems from high-order representation of data



Key ingredient:

the high-order (or generalized) Riemann problem

The high-order (or derivative, or generalized) Riemann problem:

$$\left.\begin{aligned} &\partial_t Q + \partial_x F(Q) = S(Q) \\ &Q(x,0) = \begin{cases} Q_L(x) \text{ if } x < 0 \\ Q_R(x) \text{ if } x > 0 \end{cases} \end{aligned}\right\} \quad GRP_K$$

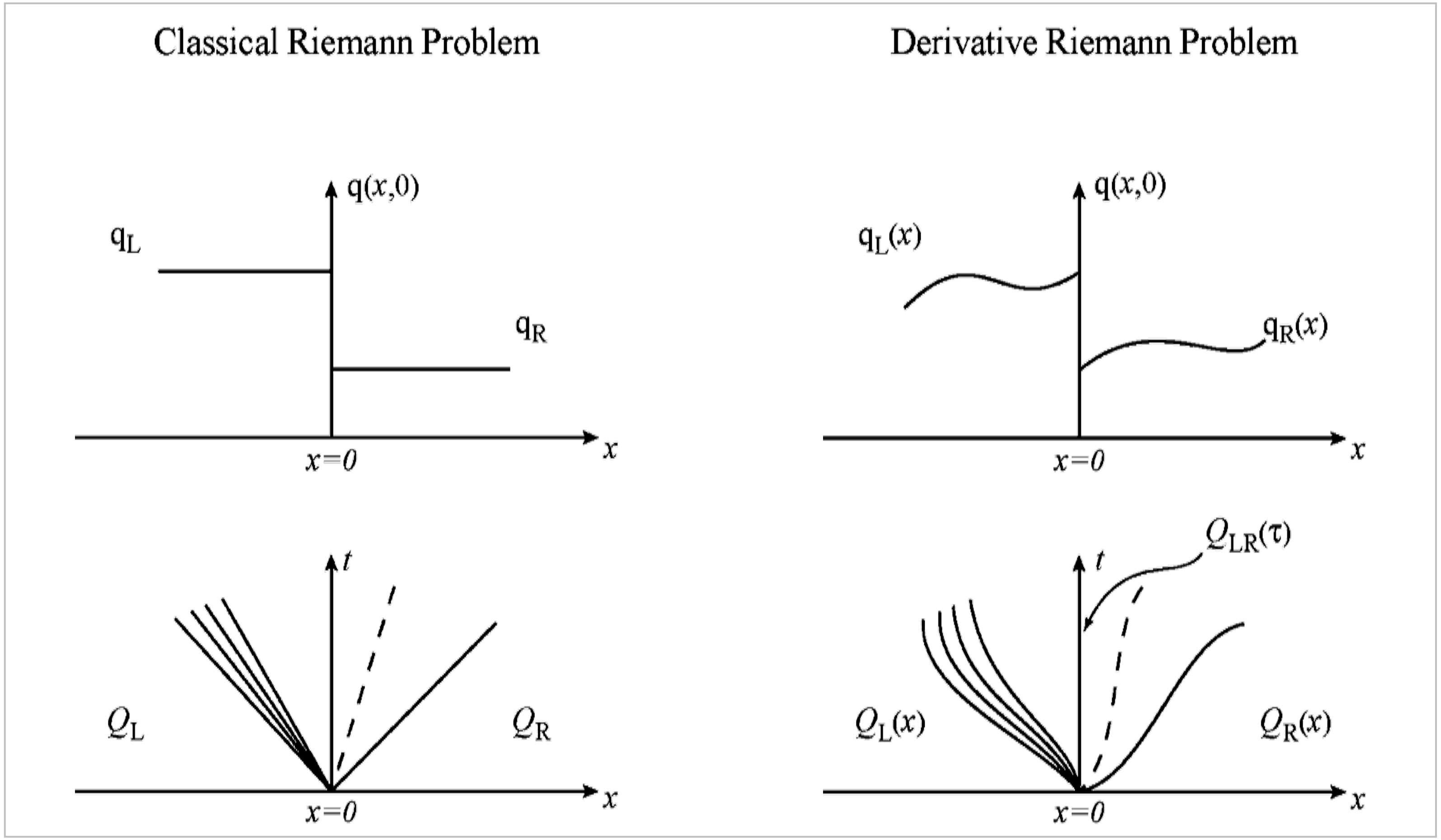
Initial conditions: two smooth functions

$$Q_L(x),\ Q_R(x)$$

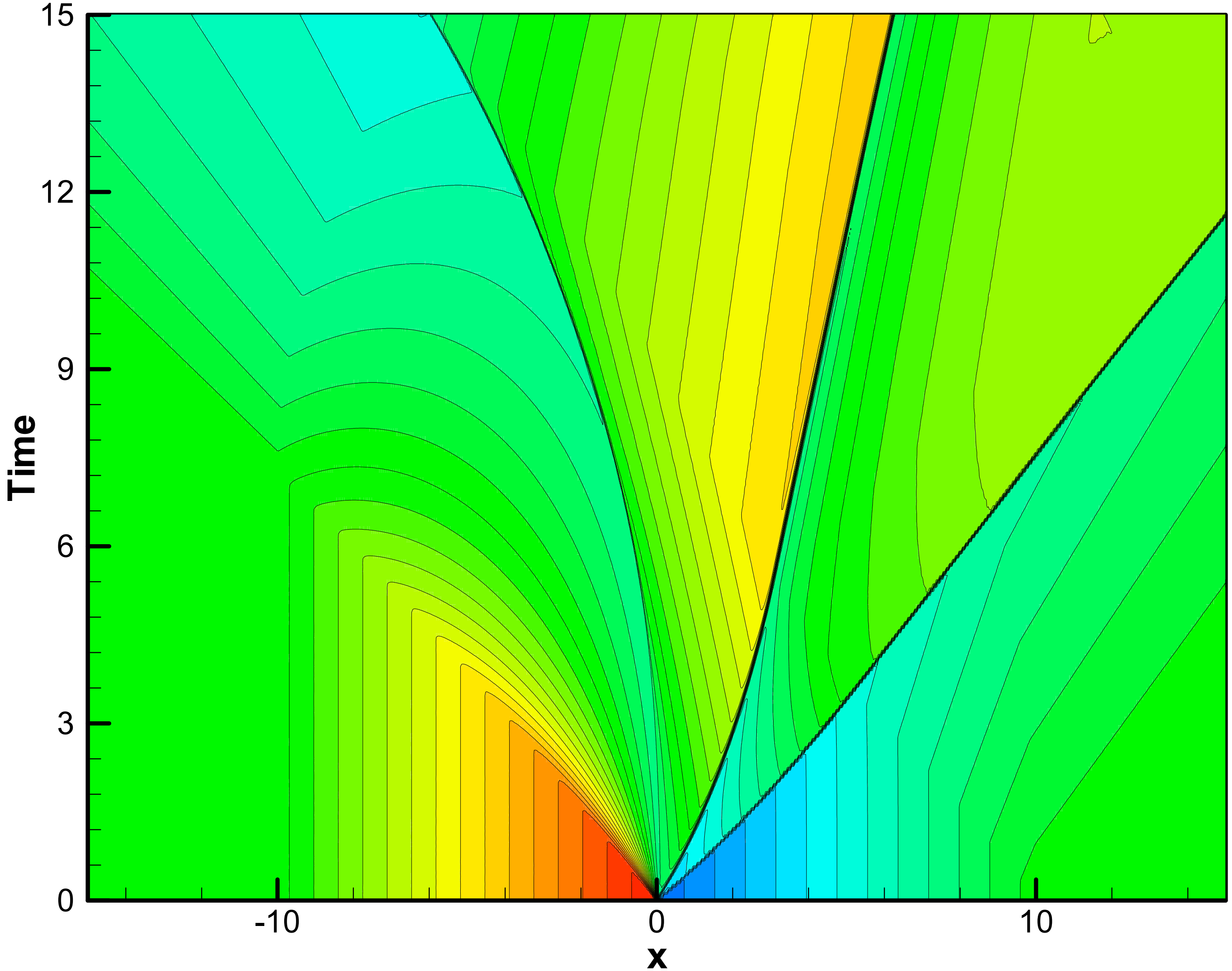
For example, two polynomials of degree K

The generalization is twofold:

(1) the intial conditions are two polynomials of arbitrary degree
(2) The equations include source terms



Classical Riemann Problem
Derivative Riemann Problem
q(x,0)
q_L
q_R
x
x=0
t
Q_L
Q_R
q(x,0)
q_L(x)
q_R(x)
x
x=0
Q_LR(τ)
t
Q_L(x)
Q_R(x)



15
12
9
6
3
0
Time
-10
0
10
x

Four solvers for the generalized Riemann problem:

E F Toro and V A Titarev. Soloution of the generalized Riemann problem for advection-reaction equations. Proc. Royal Society of London, A, Vol. 458, pp 271-281, 2002.

E F Toro and V A Titarev. Derivative Riemann solvers for systems of conservation laws and ADER methods. Journal Computational Physics Vol. 212, pp 150-165,2006

C E Castro and E F Toro. Solvers for the high-order Riemann problem for hyperbolic balance laws. Journal Computational Physics Vol. 227, pp 2482-2513,, 2008

M Dumbser, C Enaux and E F Toro. Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws . Journal of Computational Physics, Vol 227, pp 3971-4001, 2008.

Solver 1

Toro E. F. and Titarev V. A. Proc. Roy. Soc. London. Vol. 458, pp 271-281, 2002

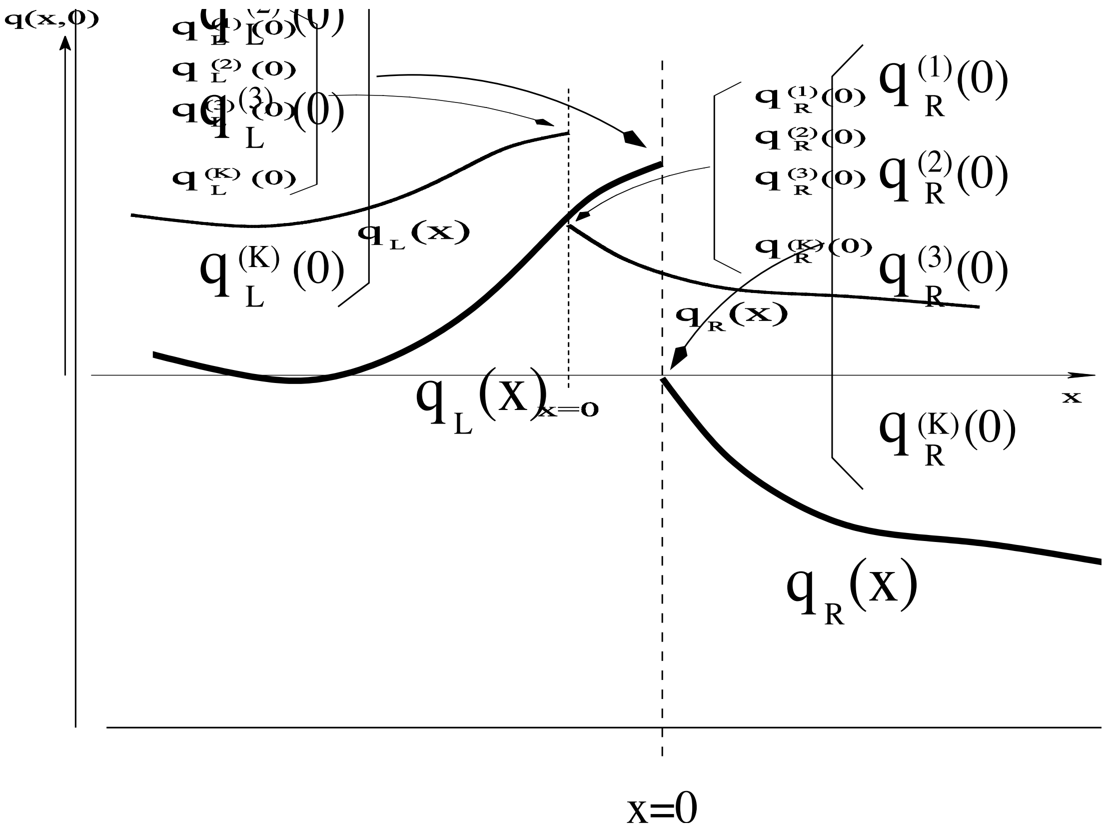
Toro E. F. and Titarev V. A. J. Comp. Phys. Vol. 212, No. 1, pp. 150-165, 2006.

$$Q_{LR}(\tau) = Q(0,0_+) + \sum_{k=1}^{K} \partial_t^{(k)} Q(0,0_+) \frac{\tau^k}{k!}$$

(Based on work of Ben-Artzi and Falcovitz, 1984, see also Raviart and LeFloch 1989)

The leading term
and
higher-order terms

Initial conditions



Computing the leading term:

Solve the *classical* RP

$$\left.\begin{aligned}&\partial_t Q + \partial_x F(Q) = 0\\ &Q(x,0) = \begin{cases} Q_L(0) & \text{if} \quad x < 0 \\ Q_R(0) & \text{if} \quad x > 0 \end{cases}\end{aligned}\right\}$$

Solution: $D^{(0)}(x/t)$

Take Godunov state at x/t=0

Leading term: $Q(0,0_+) = D^{(0)}(0)$

Computing the higher-order terms:

First use the Cauchy-Kowalewski (*) procedure yields

$$\partial_t^{(k)} Q(x,t) = G^{(k)}(\partial_x^{(0)} Q, \ldots, \partial_x^{(k)} Q)$$

Example:

$$\partial_t q + \lambda \partial_x q = 0 \Rightarrow \begin{cases} \partial_t q & = & -\lambda \partial_x q \\ \partial_t^{(2)} q & = & (-\lambda)^2 \partial_x^{(2)} q \\ \partial_t^{(m)} q & = & (-\lambda)^m \partial_x^{(m)} q \end{cases}$$

Must define spatial derivatives at x=0 for t>0

(*) **Cauchy-Kowalewski theorem. One of the most fundamental results in the theory of PDEs. Applies to problems in which all functions involved are analytic.**

Computing the higher-order terms

Then construct evolution equations for the variables:

$$\partial_x^{(k)} Q(x,t)$$

Note:

$$\partial_t q + \lambda \partial_x q = 0 \Rightarrow \partial_t(\partial_x q) + \lambda \partial_x(\partial_x q) = 0$$

For the general case it can be shown that:

$$\partial_t(\partial_x^{(k)} Q) + A(Q)\partial_x(\partial_x^{(k)} Q) = H^{(k)}(\partial_x^{(0)} Q, \partial_x^{(1)} Q, \ldots, \partial_x^{(k)} Q)$$

Neglecting source terms and linearizing we have

$$\partial_t(\partial_x^{(k)} Q) + A(Q(0,0_+))\partial_x(\partial_x^{(k)} Q) = 0$$

Computation of higher-order terms

For each k solve *classical* Riemann problem:

$$\left.\begin{aligned} &\partial_t(\partial_x^{(k)}Q) + A(Q(0,0_+))\partial_x(\partial_x^{(k)}Q) = 0 \\ &\partial_x^{(k)}Q(x,0) = \begin{cases} \partial_x^{(k)}Q_L(0) & \text{if} \quad x < 0 \\ \partial_x^{(k)}Q_R(0) & \text{if} \quad x > 0 \end{cases} \end{aligned}\right\}$$

Similarity solution $D^{(k)}(x/t)$

Evaluate solution at x/t=0

All spatial derivatives at x=0 are now defined

$$\partial_x^{(k)}Q(0,0_+) = D^{(k)}(0)$$

Computing the higher-order terms

All time derivatives at x=0 are then defined

$$\partial_t^{(k)} Q(0,0_+) = G^{(k)}(\partial_x^{(0)} Q(0,0_+),\ldots,\partial_x^{(k)} Q(0,0_+))$$

Solution of DRP is

$$Q_{LR}(\tau) = Q(0,0_+) + \sum_{k=1}^{K} \partial_t^{(k)} Q(0,0_+) \frac{\tau^k}{k!}$$

GRP-K = 1(non-linear RP) + K (linear RPs)

Options: state expansion and flux expansion

Illustration of ADER finite volume method

$$\partial_t Q + \partial_x F(Q) = S(Q)$$

Control volume in computational domain $[x_{i-1/2}, x_{i+1/2}] \times [0, \Delta t]$

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2} - F_{i-1/2}\right] + \Delta t S_i$$ **Update formula**

Integral average at time n
$$Q_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} Q(x,0)\,dx$$

Numerical flux
$$F_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} F(Q_{i+1/2}(\tau))\,d\tau$$

Numerical source
$$S_i = \frac{1}{\Delta t}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{x_{i-1/2}}^{x_{i+1/2}} S(Q_i(x,t))\,dx\,dt$$

Two more solvers are studied in:

C E Castro and E F Toro. Solvers for the high-order Riemann problem for hyperbolic balance laws. Journal Computational Physics Vol. 227, pp 2482-2513,2008

One of them is a re-interpretation of the method of Harten-Enquist-Osher-Chakravarhy (HEOC)

A. Harten, B. Engquist, S. Osher, and S.R. Chakravarthy. Uniformly high order accurate essentially non-oscillatory schemes III. *Journal of Computational Physics*, 71:231–303, 1987.

The HEOC method is in fact a generalization of the MUSCL-Hancock method of Steve Hancock (van Leer 1984)

The other solver has elements of the HEOC solver and solves linear problems for high-order time derivatives.

It is shown that all three solvers are exact for the generalized Riemann problem for a linear homogeneous hyperbolic system

The latest solver

M Dumbser, C Enaux and E F Toro. Finite volume schemes of very high order of accuracy for stiff hyperbolic balance laws. Journal of Computational Physics, Vol 227, pp 3971-4001, 2008.

Extends Harten's method (1987)**

A. Harten, B. Engquist, S. Osher, and S.R. Chakravarthy. Uniformly high order accurate essentially non-oscillatory schemes III. *Journal of Computational Physics*, 71:231–303, 1987.

- **Evolves data left and right prior to "time-interaction"**
- **Evolution of data is done numerically by an implicit space-time DG method**
- **The solution of the LOCAL generalized Riemann problem has an implicit predictor step**
- **The scheme remains globally explicit**
- **Stiff source terms can be treated adequately**
- **Reconciles stiffness with high accuracy in both space and time**

**C E Castro and E F Toro. Solvers for the high-order Riemann problem for hyperbolic balance laws. Journal Computational Physics Vol. 227, pp 2482-2513,2008

Main features of ADER schemes

One-step fully discrete schemes

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q)$$

Accuracy in space and time is arbitrary

General meshes

Unified framework

Finite volume, DG finite element and Path-conservative formulations

Main applications so far

1, 2, 3 D Euler equations on unstructured meshes
3D Navier-Stokes equations
Reaction-diffusion (parabolic equations)
Sediment transport in water flows (single phase)
Two-phase sediment transport (Pitman and Le model)
Two-layer shallow water equations
Aeroacoustics in 2 and 3D
Seismic wave propagation in 3D
Tsunami wave propagation
Magnetohydrodynamics
3D Maxwell equations
3D compressible two-phase flow, etc.

Sample results for linear advection

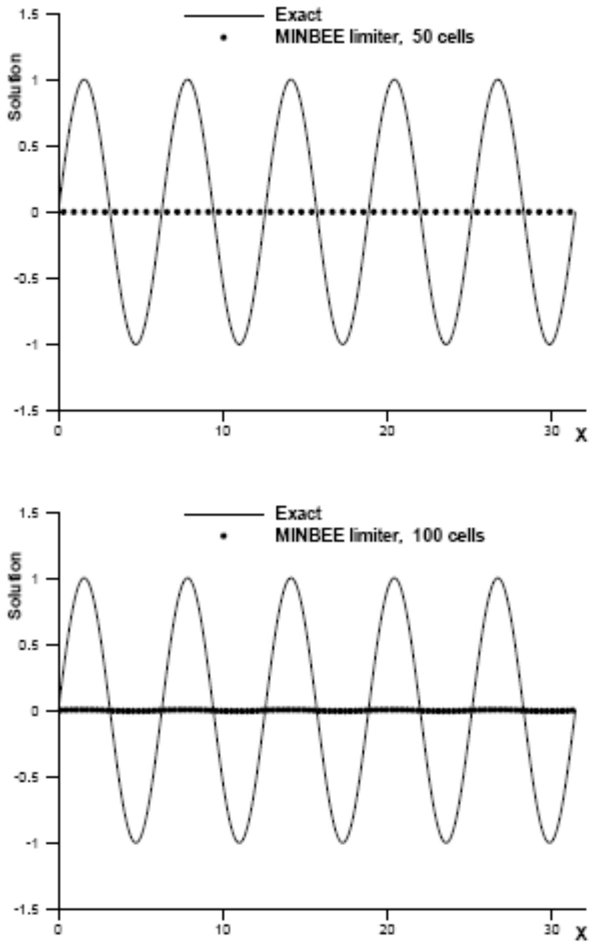


Fig. 20.2. Linear advection. Results from TVD scheme with MINBEE limiter (symbols) at time $t = 1000\pi$ using meshes of 50 and 100 cells, with $C_{cfl} = 0.95$. Exact solution shown by full line (Courtesy of Dr. V. A. Titarev).

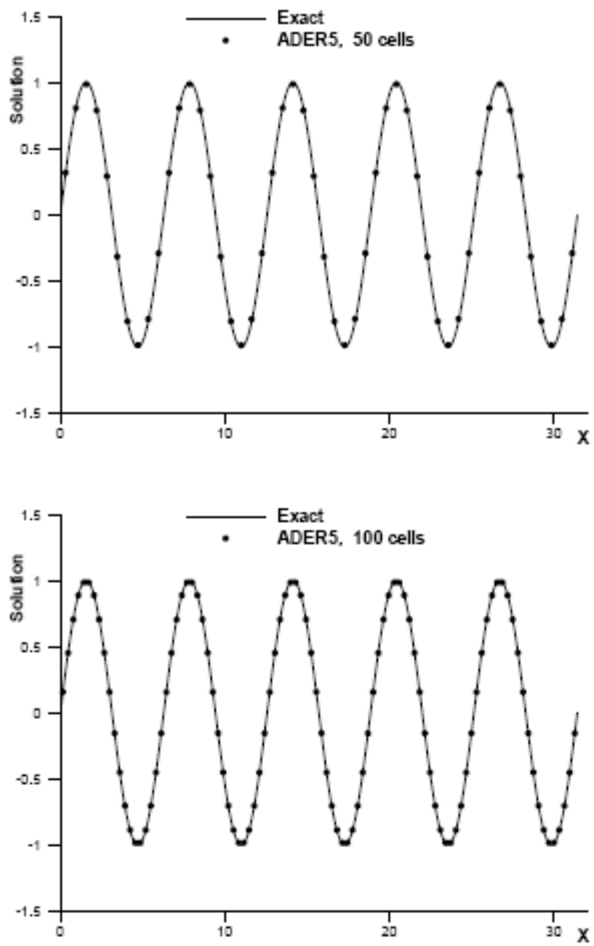
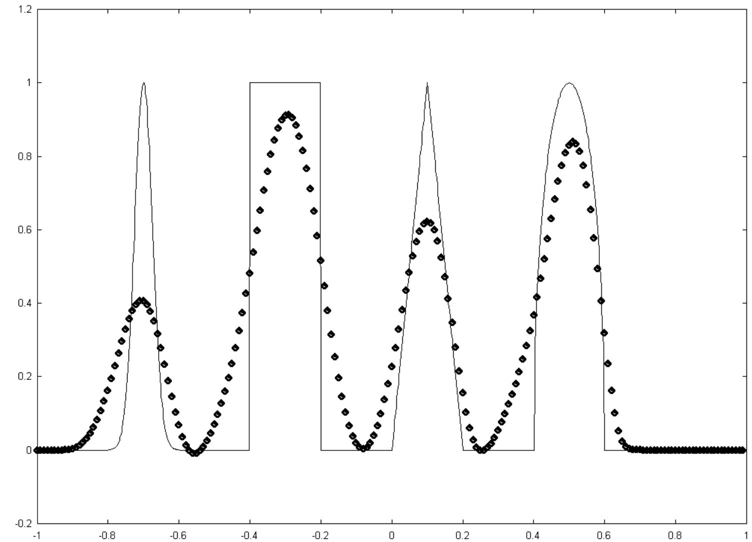
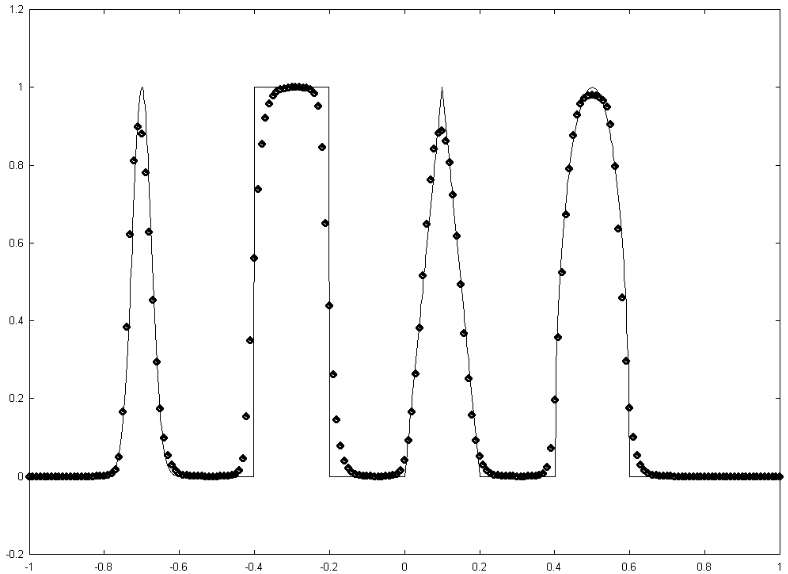


Fig. 20.4. Advection of smooth profile. Results from 5–th order ADER scheme (symbols) at time $t = 1000\pi$ using meshes of 50 and 100 cells, with $C_{cfl} = 0.95$. Exact solution shown by full line (Courtesy of Dr. V. A. Titarev).


1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

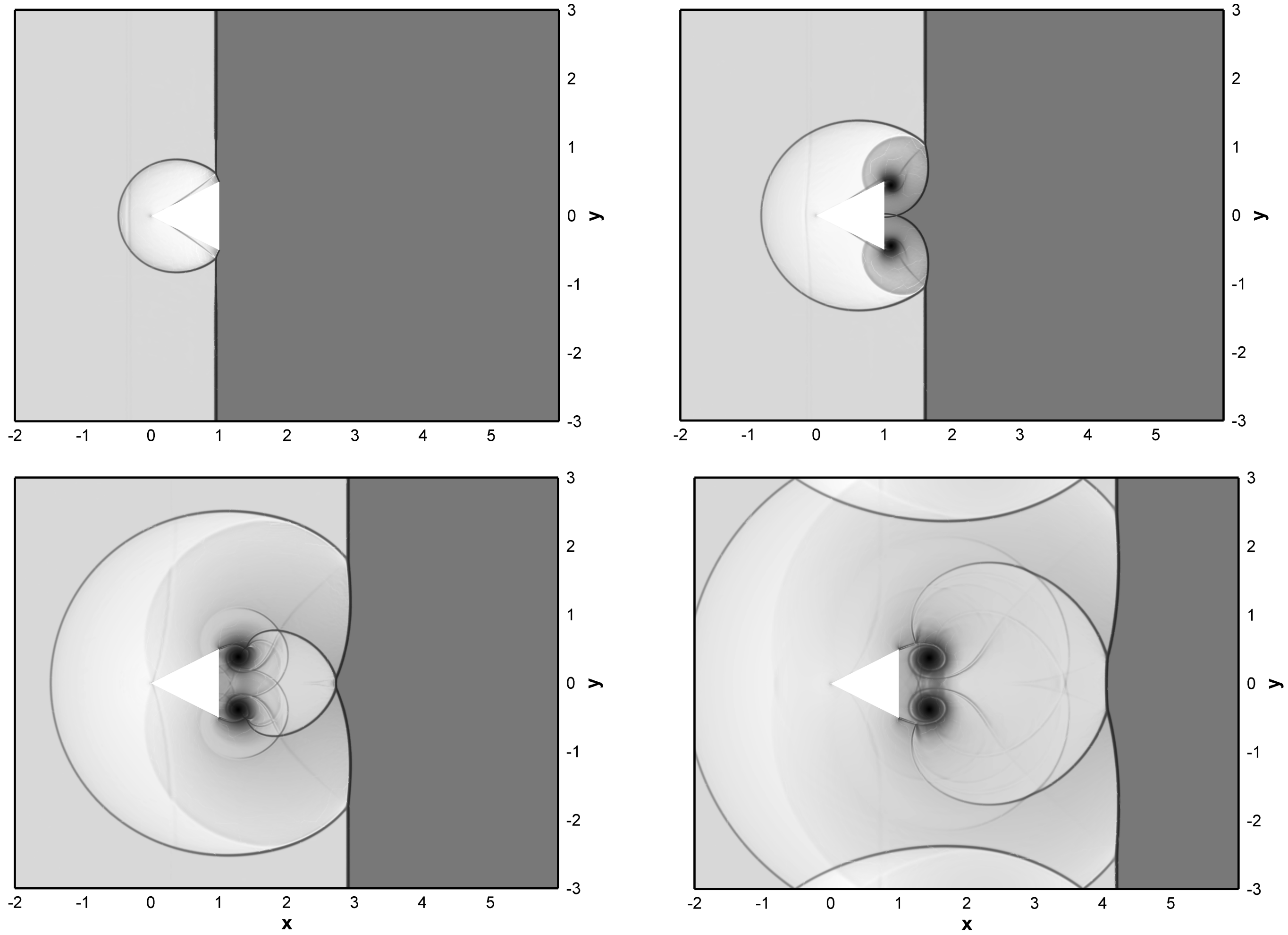
WENO-5


1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

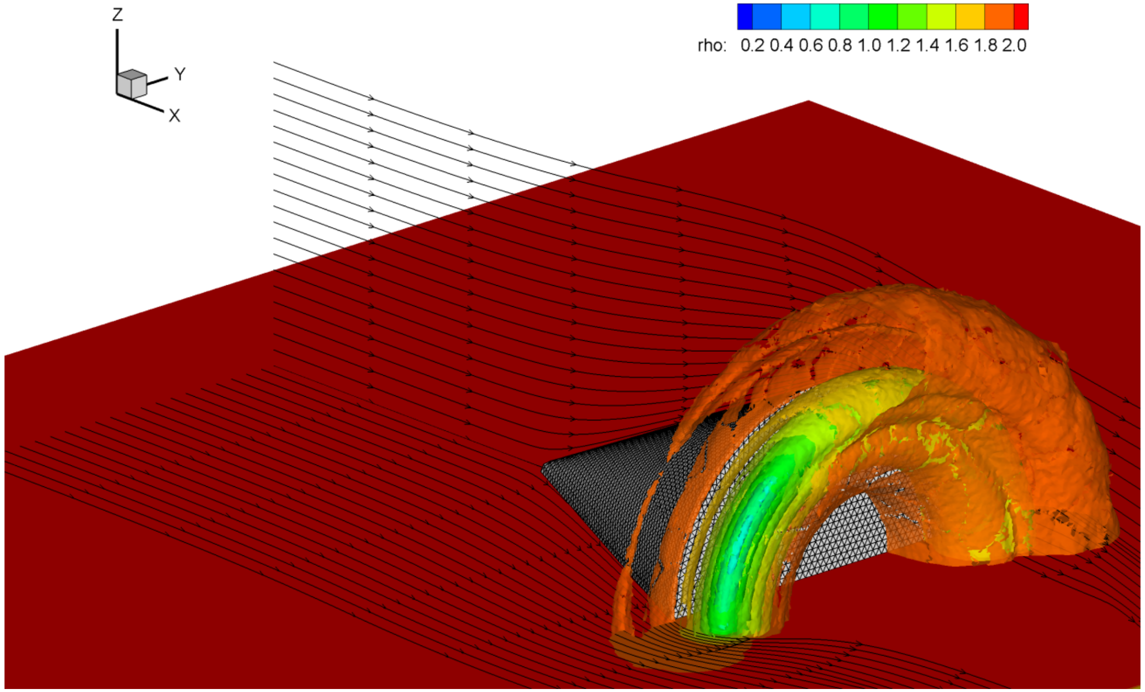
ADER-3

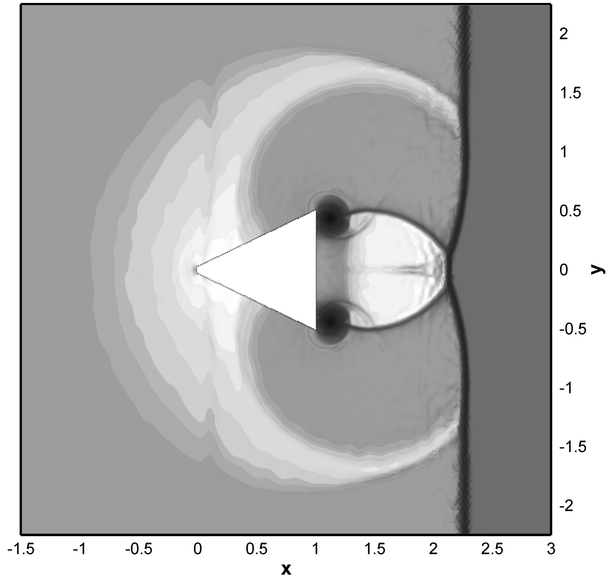
Sample results for 2D and 3D Euler equations

2D Euler equations: reflection from triangle



3D Euler equations: reflection from cone





Sample results for 2D and 3D Baer-Nunziato equations

Application of ADER to the 3D Baer-Nunziato equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}(\phi_1\rho_1) + \nabla\cdot(\phi_1\rho_1\mathbf{u_1}) = 0,\\
&\frac{\partial}{\partial t}(\phi_1\rho_1\mathbf{u_1}) + \nabla\cdot(\phi_1\rho_1\mathbf{u_1}\otimes\mathbf{u_1}) + \nabla\phi_1 p_1 = p_I\nabla\phi_1 + \lambda(\mathbf{u_2}-\mathbf{u_1}),\\
&\frac{\partial}{\partial t}(\phi_1\rho_1 E_1) + \nabla\cdot((\phi_1\rho_1 E_1 + \phi_1 p_1)\mathbf{u_1}) = -p_I\partial_t\phi_1 + \lambda\,\mathbf{u_I}\cdot(\mathbf{u_2}-\mathbf{u_1}),\\
&\frac{\partial}{\partial t}(\phi_2\rho_2) + \nabla\cdot(\phi_2\rho_2\mathbf{u_2}) = 0,\\
&\frac{\partial}{\partial t}(\phi_2\rho_2\mathbf{u_2}) + \nabla\cdot(\phi_2\rho_2\mathbf{u_2}\otimes\mathbf{u_2}) + \nabla\phi_2 p_2 = p_I\nabla\phi_2 - \lambda\,(\mathbf{u_2}-\mathbf{u_1}),\\
&\frac{\partial}{\partial t}(\phi_2\rho_2 E_2) + \nabla\cdot((\phi_2\rho_2 E_2 + \phi_2 p_2)\mathbf{u_2}) = p_I\partial_t\phi_1 - \lambda\,\mathbf{u_I}\cdot(\mathbf{u_2}-\mathbf{u_1}),\\
&\frac{\partial}{\partial t}\phi_1 + \mathbf{u_I}\nabla\phi_1 = 0.
\end{aligned}\right\} \tag{54}$$

11 nonlinear hyperbolic PDES
Stiff source terms: relaxation terms

EXTENSION TO NONCONSERVATIVE SYSTEMS:
Path-conservative schemes

DUMBSER M, HIDALGO A, CASTRO M, PARES C, TORO E F. (2009).
FORCE Schemes on Unstructured Meshes II: Nonconservative Hyperbolic Systems.
Computer Methods in Applied Science and Engineering. Online version available, 2010

Also published (NI09005-NPA) in pre-print series of the
Newton Institute for Mathematical Sciences
University of Cambridge, UK.

It can be downloaded from
http://www.newton.ac.uk/preprints2009.html

CASTRO M, PARDO A, PARES C, TORO E F (2009).
ON SOME FAST WELL-BALANCED FIRST ORDER SOLVERS FOR NONCONSERVATIVE SYSTEMS.
MATHEMATICS OF COMPUTATION. ISSN: 0025-5718. Accepted.

Three space dimensions

Unstructured meshes

Path-conservative method

Centred non-conservative FORCE is bluilding block

ADER: high-order of accuracy in space and time

(implemented upto 6-th order in space and time)

Reference solutions to the BN equations

Exact Riemann solver of Schwendemann et al. (2006) (1D)

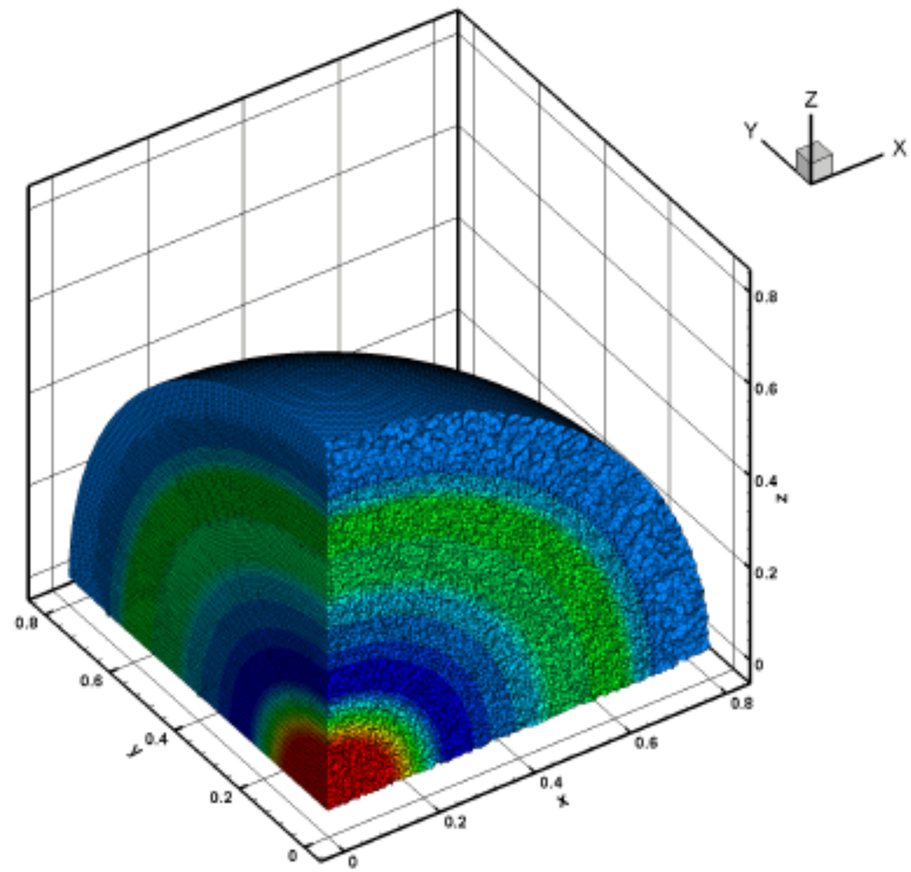
Exact smooth solution the 2D BN equations to be used in convergence rate studies (Dumbser et al. 2010)

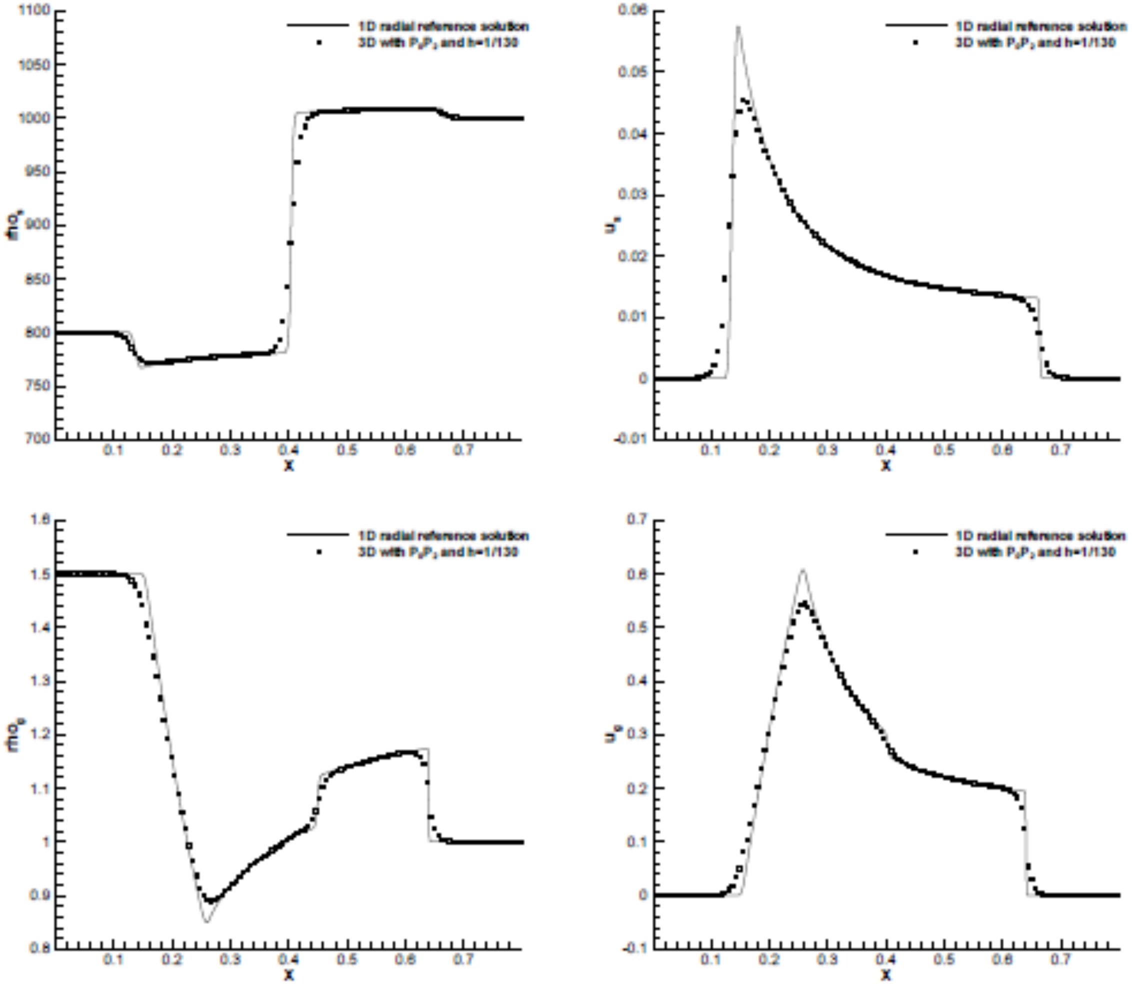
Spherically symmetric 3D BN equations reduced to 1D system with geometric source terms. This is used to test 2 and 3 dimensional solutions with shocks (Dumbser et al. 2010)

Convergence rates study in 2D unstructured meshes

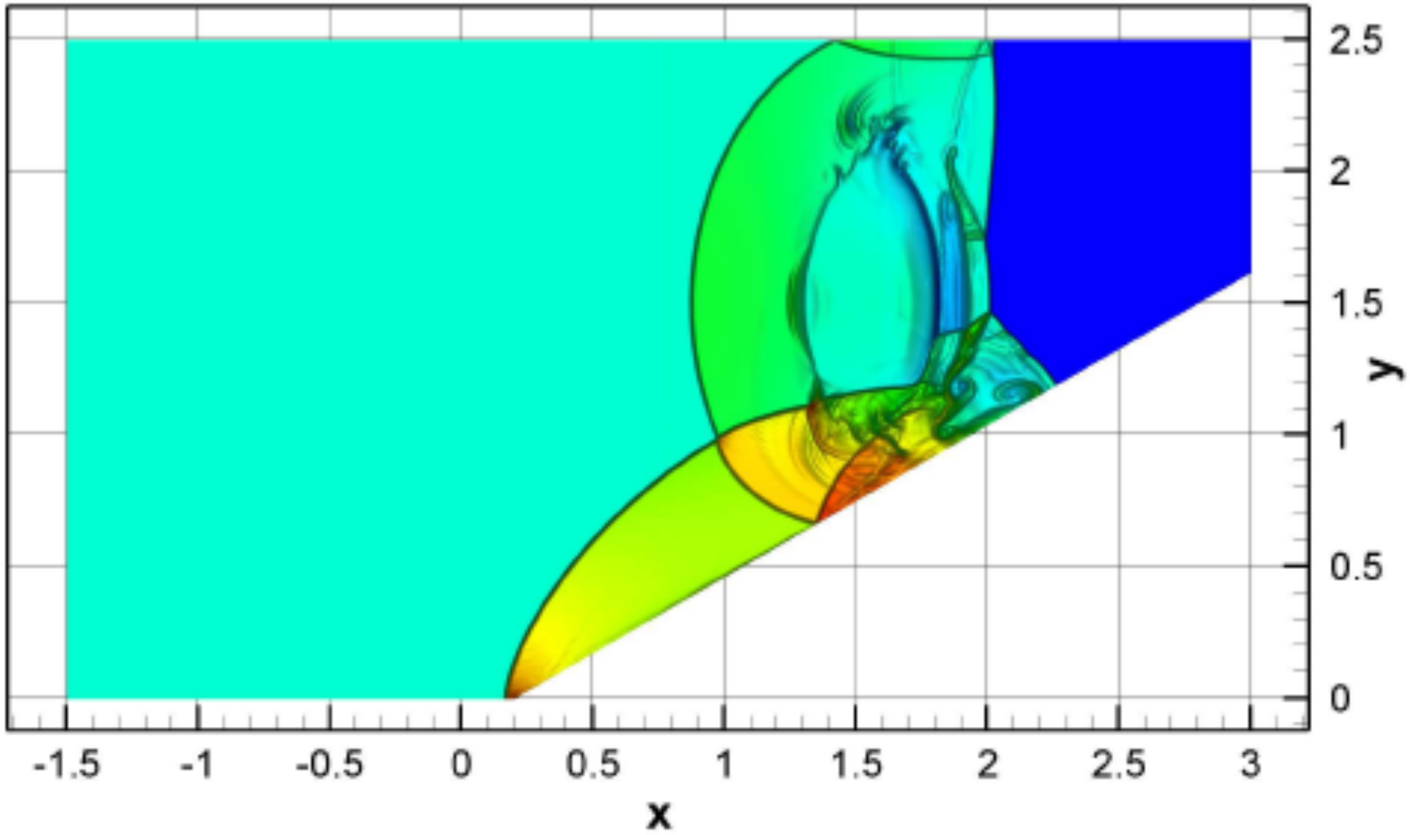
N_G	L^2	$\mathcal{O}_{L^2}$	L^2	$\mathcal{O}_{L^2}$	L^2	$\mathcal{O}_{L^2}$	L^2	$\mathcal{O}_{L^2}$	L^2	$\mathcal{O}_{L^2}$	L^2	$\mathcal{O}_{L^2}$
$\mathcal{O}2$	P_0P_1		P_1P_1									
64/24	1.86E-01		2.04E-01									
128/48	5.94E-02	1.7	3.04E-02	2.7								
192/64	2.80E-02	1.9	1.45E-02	2.6								
256/128	1.75E-02	1.6	1.92E-03	2.9								
$\mathcal{O}3$	P_0P_2		P_1P_2		P_2P_2							
32 /16	5.09E-01		2.77E-01		5.59E-02							
64 /24	1.63E-01	1.6	8.97E-02	2.8	1.67E-02	3.0						
128/32	3.50E-02	2.2	2.91E-02	3.9	6.56E-03	3.2						
192/64	1.16E-02	2.7	2.07E-03	3.8	7.84E-04	3.1						
$\mathcal{O}4$	P_0P_3		P_1P_3		P_2P_3		P_3P_3					
32 /16	1.71E-01		1.95E-01		2.14E-02		1.77E-02					
64 /24	1.71E-02	3.3	4.95E-02	3.4	3.79E-03	4.3	2.46E-03	4.9				
128/32	1.28E-03	3.7	1.45E-02	4.3	8.95E-04	5.0	5.61E-04	5.1				
192/64	2.80E-04	3.7	5.16E-04	4.8	3.94E-05	4.5	2.07E-05	4.8				
$\mathcal{O}5$	P_0P_4		P_1P_4		P_2P_4		P_3P_4		P_4P_4			
32 /16	2.09E-01		9.85E-02		9.70E-03		5.22E-03		1.79E-03			
64 /24	2.30E-02	3.2	1.75E-02	4.3	1.18E-03	5.2	5.56E-04	5.5	2.24E-04	5.1		
128/32	1.16E-03	4.3	3.27E-03	5.8	2.09E-04	6.0	8.36E-05	6.6	4.36E-05	5.7		
192/64	1.63E-04	4.8	4.53E-05	6.2	7.23E-06	4.9	2.28E-06	5.2	1.75E-06	4.6		
$\mathcal{O}6$	P_0P_5		P_1P_5		P_2P_5		P_3P_5		P_4P_5		P_5P_5	
32 / 8	8.45E-02		5.50E-01		1.49E-01		6.22E-02		5.90E-02		2.76E-02	
64 /16	3.09E-03	4.8	8.72E-02	2.7	5.90E-03	4.7	1.73E-03	5.2	6.12E-04	6.6	4.69E-04	5.9
128/24	5.95E-05	5.7	1.46E-02	4.4	6.18E-04	5.6	1.39E-04	6.2	4.18E-05	6.6	3.72E-05	6.2
192/32	5.39E-06	5.9	2.39E-03	6.3	8.31E-05	7.0	2.17E-05	6.5	5.12E-06	7.3	4.99E-06	7.0

BN equations: spherical explosion test

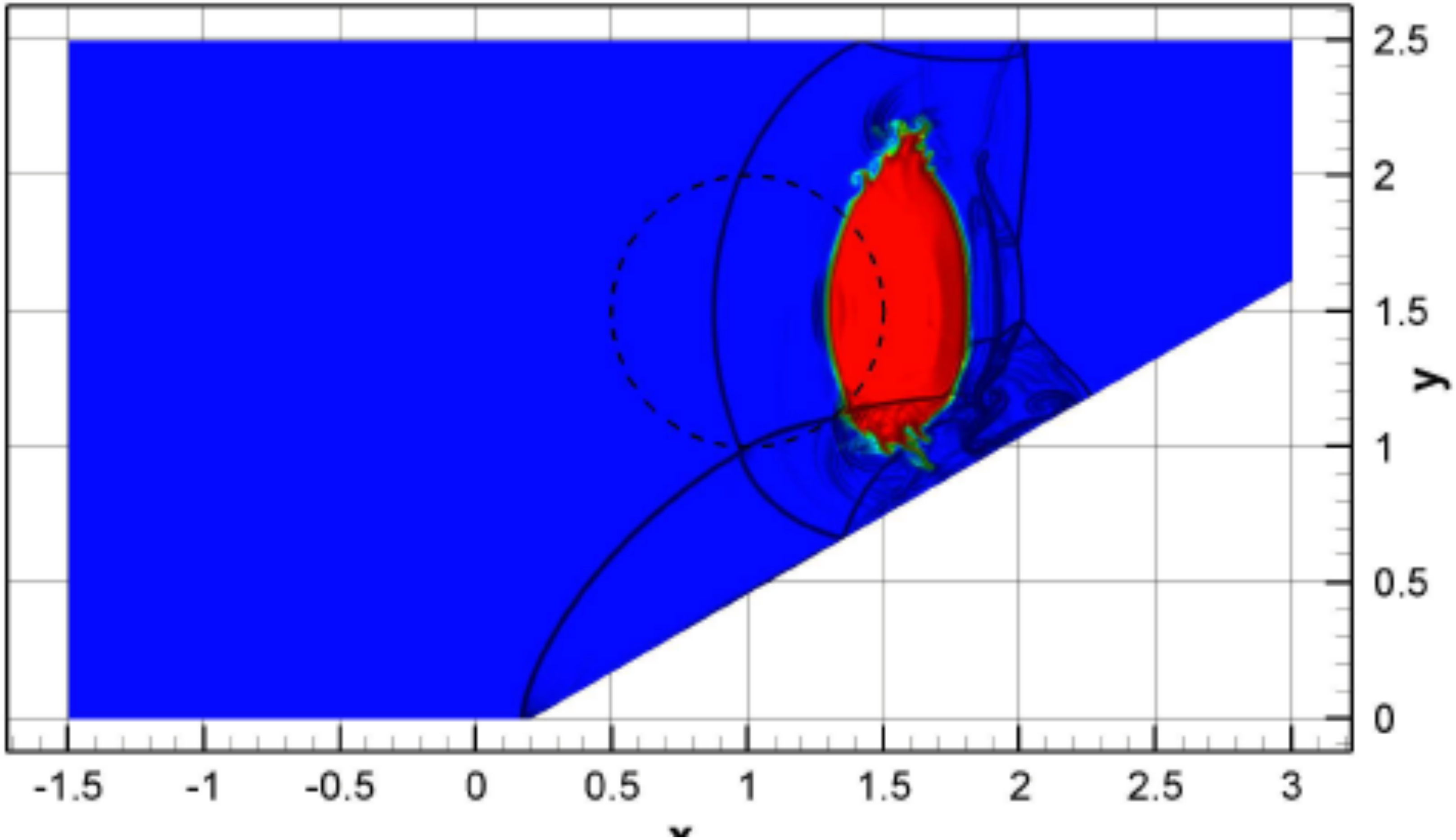



1D radial reference solution
3D with P0P2 and h=1/130
rho_s
u_s
rho_g
u_g
x

Double Mach reflection for the 2D Baer-Nunziato equations



Double Mach reflection for the 2D Baer-Nunziato equations



Further reading:

Chapters 19 and 20 of:

Toro E F. Riemann solvers and numerical methods for fluid dynamics.
Springer, Third Edition, 2009.

Thank you