The dynamics of shallow fluid flows: Modeling and numerical analysis

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Abstract

Shallow water models are an efficient simplification of the incompressible Navier-Stokes equations. They are widely used in hydraulic engineering and geophysics. These notes treat the derivation and numerical discretization of shallow water models. A key point of the numerical analysis is the well-balancing of equilibrium solutions like lakes and rivers. Besides the standard shallow water equation we also derive multi-layer models in detail and discuss the possible loss of hyperbolicity. Special attention is paid to the non-conservative nature of shallow water models, and a new approach to recover conservative numerical fluxes. Finally, we discuss a class of viscous shallow water models which hopefully will extend the stability region of depth-integrated multi-layer models.

Contents

1 Introduction

This is an announcement for the forthcoming lecture series given at the NSPDE, Malaga, February 2010. Details will be worked out for the final lecture notes.

My lectures will be concerned with the mathematical modeling and numerical analysis of free-surface waterflows as they occur in rivers, lakes, oceans or channels. The fundamental model is given by the incompressible Navier-Stokes equations. There are several major challenges concerning their discretization:

- a three-dimensional spatial domain with a time-dependent free surface, which requires techniques like a moving grid or a level-set method
- an elliptic constraint for the pressure which implies infinite propagation speeds and demands an implicit solver.

In many cases of practical interest the incompressible Navier-Stokes equations can be simplified considerably. Under the assumptions of a hydrostatic pressure and a depth-independent horizontal velocity profile, depth-integration of the Navier-Stokes equations leads to the shallow water equations. Several features make this system very efficient:

- the three-dimensional domain with moving free-surface is replaced by a two-dimensional, fixed domain
- the free surface is replaced by a simple height variable whose evolution is governed by a continuity equation
- the shallow water equations are a hyperbolic system of conservation laws (often with additional source terms modeling topography, coriolis forces, bottom friction ...). Thus there is no elliptic constraint any more, and the speed of propagation is the finite speed of surface waves.
- the numerical techniques of conservative finite difference, finite volume and discontinuous Galerkin methods can be carried over from gas dynamics, leading to fast and robust solvers.

These notes treat the derivation and numerical discretization of shallow water models. In Section 2 we discuss the assumptions under which depth-integration leads to the inviscid shallow water model. In Section 3 we discuss the difficulties in computing solutions which are close to equilibrium, and introduce well-balanced schemes for one-dimensional moving water flows. In Section 4 we derive the multi-layer shallow water equations for layers of different densities. In particular, we show that hyperbolicity of the equations can break down when shear velocities are not small. In Section 5 we discuss non-conservative aspects of the depth-integrated models, in particular difficulties to define weak solutions, Roe-matrices, and uniqueness. We present a new approach to recover conservative numerical fluxes for these models. Finally, in Section 6 we introduce slightly more general assumptions on the vertical velocity profiles leading to viscous models. We hope that these models will be stable for a larger class of multi-layer flows under shear.

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2 Derivation of the Inviscid Shallow Water Equations

This section treats the derivation of the simplest inviscid shallow water model. At the same time, it prepares the techniques which we use in later sections to derive more sophisticated models.

2.1 The 3D Navier-Stokes Equations

We start from the incompressible Navier-Stokes equations together with kinematic boundary condition at the bottom and at the free surface. In order to include density layered flows in Section 4 we admit water of variable density.

2.2 Non-Dimensionalization

Here we write the conservation laws of mass and momentum (which are part of the incompressible Navier-Stokes equations) in dimensionless form. This reveals the water depth ε and the Froude number $\mathbf{Fr}$ as crucial dimensionless numbers.

2.3 Hydrostatic Assumption

In the limit of small $\varepsilon^2\mathbf{Fr}^2$ and $\nu\mathbf{Fr}^2$ the equation of vertical momentum reduces to the hydrostatic pressure law. In other words, the assumption that the pressure is hydrostatic arises naturally in this limit. This pressure law is then used in the equations of horizontal momentum.

2.4 Depth Integration

A main simplification will be in the velocity field. While the Navier-Stokes equations model a fully three-dimensioal field $(u, v, w)(x, y, z, t)$, the shallow water model model the depth-integrated field $(\bar{u}, \bar{v})(x, y, t)$. Roughly speaken, the vertical velocity w can be recovered by the kinematic boundary condition and the incompressibility condition $\mathbf{div}(u, v, w) = 0$. The z-dependence of the variables is eliminated by depth integration.

2.5 Shallow Water Assumption

The assumption of hydrostatic pressure and depth integration give evolution equations for the depth integrated mass and momentum. However, the convective flux of the momentum equation contains a vertical integral of ρu^2 which cannot be expressed in terms of integrated mass or momentum. It is here that the most far-reaching assumption is used: one assumes that the horizontal velocity field (u, v) is independent of the vertical variable z, i.e $(u, v)(x, y, z, t) = (u, v)(x, y, t)$. We are using this assumption here, but we will replace it by less restrictive assumptions Section 6.

3 High-Order Well-Balancing for Moving Equilibria

Over the last decade, the development of high resolution *well-balanced* schemes was a central topic in the numerical analysis of hyperbolic systems. Indeed, many applications in continuum mechanics lead to systems of balance laws. For such flows, the source terms are often in near-perfect equilibrium with the convective forces. A numerical scheme which does not respect these equilibria at the discrete level may produce spurious oscillations, and hence convergence may slow down.

3.1 Classes of equilibria

Most numerical approximations are only well-balanced for certain subclasses of equilibria. Often, these stationary states are characterized by constant momentum and energy, together with further simplifying assumptions. The more complex the equilibria are, the more involved well-balancing becomes. For example, compared with balancing still water (see e.g. [3]), the moving water case solved in [19] was quite a technical challenge. More and more, these publications are only accessible to a small circle of experts.

3.2 Ingredients of a unifying framework

Therefore, we see a strong need for a unifying and at the same time simplifying framework within which existing schemes may be rederived and reinterpreted, and new ones may be developed more easily. Here we will present such a framework for finite volume and discontinuous Galerkin schemes.

When deriving a well-balanced scheme, we first select a class of stationary states which the scheme should preserve. Examples are the lake at rest, river flows, jets in a rotational frame, or multi-layer shallow water flows. For each component of the algorithm, i.e.

- reconstruction
- quadrature
- flux-source-computation at the cell interface

we define appropriate notions of well-balancing, always tailored to the class of stationary states under consideration. Once these three balancing-properties are fulfilled, well-balancing of the overall scheme follows immediately. As an example, we demonstrate that the scheme based on hydrostatic reconstruction of Audusse et al. [3], the scheme based on a more general equilibrium reconstruction by Noelle, Xing and Shu [19] and the schemes based on a general hydrostatic reconstruction by Castro, Pares et al [7] fall into this framework.

4 Multi-Layer Systems

In this section we go beyond single layer flows and consider stratified water flows consisting of two or more layers. These layers are stabilized by differences in salinity and/or temperature, which act as barriers to mixing. An example which has recently found intense attention in applied mathematics and numerical analysis is the superposition of the Mediterranean and the Atlantic in the Strait of Gibraltar, cf. e.g. Castro, Macias and Pares [10]. The layering persists as long as the velocity difference (shear velocity) is not too large.

4.1 Derivation

A shallow water multi-layer system can be derived by depth-integrating the incompressible Navier-Stokes equations over each layer. The multi-layer system resembles a set of coupled shallow water models. If the shear velocities are not too large, and if the density differences are big enough, these equations form a non-conservative hyperbolic system in one or two space dimensions. Numerically, this system can be solved efficiently and accurately by well-balanced finite volume schemes, cf. e.g. Castro, Gallardo and Pares [8].

4.2 Loss of Hyperbolicity

As indicated, the layering may become unstable. A real world experiment takes place daily in the Strait of Gibraltar. In regions of strong gradients in the bottom topography, the lower layer mediterranean water accelerates, causing a strong shear and hence a Kelvin-Helmholtz instability. This leads to a local mixing of the layers. Mathematically, the hyperbolicity is lost, and usually the finite volume solvers break down.

4.3 Possible stabilizations

Once the multi-layer models break down, the most thorough, but expensive, modeling solution would be to replace the shallow water model by the full 2D or 3D Navier-Stokes equations and to apply the respective incompressible solvers.

A challenging question is whether one can model such local mixing without giving up the depth-averaging everywhere. Several authors have suggested such models. Kim and LeVeque [14] as well as Castro and Pares [11] add friction to recover hyperbolicity. Bouchut and Morales [4] treat the loss of hyperbolicity by a numerical splitting. Audusse [2] (who is not focussing on stratified flows) treats the elliptic terms as source terms and integrates them implicitly. In [6] we have explored an adaptive two/three layer model. When hyperbolicity breaks down, we introduced a third layer near the interface, which realizes the mixing instantaneously. As discussed in the same paper, this approach is of limited success: for many non-hyperbolic two-layer states we could not find a corresponding three-layer state which would be hyperbolic.

5 Conservative and Non-Conservative Aspects of Shallow Water Systems

A main shortcoming of the analytical structure of the shallow water equations is that for varying topography, or for multi-layer flows, the shallow water equations are not conservative. This may be surprising, because the equations of continuity and momentum of the incompressible Navier-Stokes equations, from which the shallow water equations are derived, are conservative.

5.1 Mathematical difficulties related to the loss of conservation

Once conservation is lost, we lack structure to define the speeds of discontinuous fronts. We also loose Lax' solution of the Riemann-problem, which forms the basis of the most successful numerical solvers. In some cases, nonunique solutions could be constructed (see e.g. [1]).

5.2 Path conservative schemes

In 1995, Dal Maso, Murat and LeFloch [12] introduced a theory of paths in state space, along which conservativity is recovered. In a series of works, Castro and Pares (see eg. [5]) used this theory to build high-order accurate, well-balanced, path-conservative Roe-type solvers for multi-layer flows. It should be noted, however, that the choice of path is not unique, and natural choices of paths are still being developed. For a detailed analysis of the effect of the convergence error due to the non-conservative term see [9].

5.3 Recovering Conservative Fluxes

The present contribution is a first step towards establishing an alternative route to discretize these non-conservative systems. When we integrate the momentum equation of the Navier-

Stokes system vertically, and before we combine all the resulting three-dimensional fluxes into a single horizontal shallow water flux, we can distinguish two types of flux-terms: the first is a gravitational pressure term which acts upon the interfaces of the multi-layer flow. The second consists of the depth-integrated hydrodynamic flux acting horizontally within a single fluid layer. The structure of these two flux-terms is as follows:

- The interface pressure (the weight of the fluid above the interface) is not in conservation form with respect to the horizontal variables. However, it is by construction continuous in vertical direction, and therefore, it can be evaluated directly.
- The depth-integrated hydrodynamic flux may be discontinuous, but it is always in conservation form. Therefore, it can be evaluated by an approximate Riemann solver.

Together, this leads to a straightforward derivation of a simple multi-layer flow solver.

6 Derivation of Extended Shallow Water Models

A prize which we have to pay for the efficiency and simplicity of the shallow water equations is a restrictive ansatz for the vertical structure of the horizontal velocity profile. In the previous section, we chose $\partial_z(u,v)(x,y,z) \equiv 0$, so all vertical effects were neglected. There were no boundary layers, no vertical mixing, and there was no viscosity. In this section, we will experiment with two alternative assumptions. The first is a laminar boundary layer, which we expect for slow flows. The second is a turbulent boundary layer.

6.1 Laminar Boundary Layer Assumption

The starting point here are Prandtl's boundary layer equations and Blasius' solutions of flow over a flat plate. Further approximate solutions still need to be identified.

6.2 Turbulent Boundary Layer Assumption

Our research plan here is to identify relevant special solutions of a turbulence model, and to derive an ansatz for a vertival velocity profile and for an effective viscosity coefficient from such solutions.

6.3 Modified Convective Fluxes and Viscous Fluxes

Any non-constant ansatz for the vertical profile of the horizontal velocity leads immediately to a correction of the convective flux.

Similarly, a non-constant vertical velocity profile leads to a contribution in the viscous flux, which enters the shallow water momentum equation as a zero-order stabilizing source term (see e.g. [13]).

Once relevant boundary layers have been identified, there is hope that the new extended shallow water system will indeed remain hyperbolic whenever the interfaces are physically stable.

7 Summary

The shallow water equations are a widely used and in many cases extremely efficient model for free surface flows. The aim of the project sketched in these notes is to extend the range of applicability of shallow water models.

We have derived the equations in full detail, and presented a recent high order well-balanced code for rather general equilibria. From then on, the lecture covered work in progress. For multi-layer flows, hyperbolicity may break down. Non-conservativity makes it challenging to discretize the coupling terms, but we have formulated a new way to tackle this problem. A new modeling ingredient is the inclusion of more sophisticated assumptions upon the vertical structure of the horizontal velocity field. In particular, this introduces stabilizing first order terms into the momentum equations which are the vertical integrals of the viscous Navier-Stokes fluxes.

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