The Dynamics of Shallow Fluid Flows: Modeling and Numerical Analysis

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- incompressible Navier-Stokes equations
  - 2D, 3D
  - free surface
  - conservative
  - hyperbolic/parabolic/elliptic
- shallow water models
  - 1D, 2D
  - continuity equation
  - non-conservative
  - hyperbolic
- standard model in hydraulic engineering, geophysics

1. Inviscid Shallow Water Equations
2. High-Order Well-Balancing for Moving Equilibria
3. Multi-Layer Systems
4. Conservative and Non-Conservative Aspects
5. Extended Shallow Water Models

Chapter 1

Inviscid Shallow Water Equations

$$\begin{aligned} b(x),\ \eta(x,t) &\quad \textbf{bottom, surface} \\ \Omega := \{\,(x,z,t)\ \mid\ b(x) < z < \eta(x,t)\,\} &\quad \textbf{domain} \\ \rho(x,z,t) &\quad \textbf{density} \\ (u,w)(x,z,t) &\quad \textbf{velocities} \end{aligned}$$

conservation of mass (continuity equation)

$$\partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0$$

conservation of momentum (Newton's law)

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho u w) = -\partial_x p \qquad + \partial_x \sigma_{xx} + \partial_z \sigma_{xz}$$

$$\partial_t(\rho w) + \partial_x(\rho u w) + \partial_z(\rho w^2) = -(\partial_z p + \rho g) + \partial_x \sigma_{zx} + \partial_z \sigma_{zz}$$

incompressibility

$$\partial_x u + \partial_z w = 0$$

Cauchy Stress Tensor:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix} = (\lambda+\mu)\,(\partial_x u+\partial_z w)\,\mathrm{Id} + \mu \begin{pmatrix} 2\partial_x u & \partial_z u+\partial_x w \\ \partial_z u+\partial_x w & 2\partial_z w \end{pmatrix}$$
$$= \mu \begin{pmatrix} 2\partial_x u & \partial_z u+\partial_x w \\ \partial_z u+\partial_x w & 2\partial_z w \end{pmatrix}$$

λ first Lamé coefficient
μ second Lamé coefficient (**dynamic viscosity**)

Tangential flow at top and bottom:

$$w = \frac{D}{Dt}\eta = \partial_t \eta + u\partial_x \eta \qquad \text{for} \quad z = \eta(x,t)$$
$$w = \frac{D}{Dt}b = \partial_t b + u\partial_x b \qquad \text{for} \quad z = b(x)$$

Newton's law rewritten:

Advection, **pressure gradient and gravity**, **viscous forces**

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho uw) = -\partial_x p \qquad + \mu \triangle u$$

$$\partial_t(\rho w) + \partial_x(\rho uw) + \partial_z(\rho w^2) = -(\partial_z p + \rho g) + \mu \triangle w$$

space, time, density, velocities:

$$(x_{ref}, z_{ref}, t_{ref}), \quad (\rho_{ref}, u_{ref}, w_{ref})$$

pressure:

$$p_{ref} = g\rho_{ref} z_{ref}.$$

Dimensionless Numbers

ε	$= x_{ref}/z_{ref}$	**shallow water**
F	$= u_{ref}/\sqrt{g z_{ref}}$	**Froude number**
ν	$= \mu/(\rho_{ref} u_{ref} x_{ref})$	**dimensionless viscosity**

Tidal flow, Strait of Gibraltar:

$$
\begin{array}{rcllll}
z_{ref} & = & 2.0 \cdot & 10^2 & m & \\
t_{ref} & = & 2.0 \cdot & 10^4 & s & \text{(6 hours)} \\
u_{ref} & = & 1.0 & & m/s & \\
x_{ref} = t_{ref}\, u_{ref} & = & 2.0 \cdot & 10^4 & m & \\
g_{ref} & = & 9.8 & & m/s^2 & \text{(reference gravity)} \\
\rho_{ref} & = & 1.0 \cdot & 10^3 & kg/m^3 & \\
\mu & = & 1.5 & & kg/s & \text{(water } 5^0 \text{ Celsius)}
\end{array}
$$

$\Rightarrow$

$$
\begin{array}{rcll}
\varepsilon^2 & = & 1.0 \cdot & 10^{-4} \\
F^2 & = & 5.1 \cdot & 10^{-4} \\
\nu & = & 7.5 \cdot & 10^{-8}
\end{array}
$$

$\Rightarrow$

$$
\nu \ll F^2 \approx \varepsilon^2 \ll 1
$$

Dimensionless variables:

$$(\widehat{x},\, \widehat{z},\, \widehat{t}) = (x/x_{ref},\, z/z_{ref},\, t/t_{ref})$$

$$(\widehat{\rho},\, \widehat{u},\, \widehat{w}) = (\rho/\rho_{ref},\, u/u_{ref},\, w/w_{ref})$$

$$\widehat{p} = p/p_{ref}$$

- rewrite Navier-Stokes equations in dimensionless variables
- drop the “hat”: $(\widehat{x},\, \widehat{z},\, \widehat{t} \ldots \widehat{p}) \rightsquigarrow (x,\, z,\, t \ldots p)$

dimensionless conservation of momentum

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho u w) = -\frac{1}{F^2}\partial_x p + \nu\left(\partial_{xx}u + \frac{1}{\varepsilon^2}\partial_{zz}u\right)$$

$$\partial_t(\rho w) + \partial_x(\rho u w) + \partial_z(\rho w^2) = -\frac{1}{\varepsilon^2 F^2}(\partial_z p + \rho) + \nu\left(\partial_{xx}w + \frac{1}{\varepsilon^2}\partial_{zz}w\right)$$

replace conservation of vertical momentum

$$\partial_z p + \rho \; = \; -\,\varepsilon^2\, F^2 \left(\partial_t(\rho w) + \partial_x(\rho u w) + \partial_z(\rho w^2)\right)$$
$$+\,\nu\, F^2 \left(\varepsilon^2 \partial_{xx} w + \partial_{zz} w\right)$$

by the hydrostatic assumption

$$\partial_z p + \rho = 0$$

i.e.

$$p(z) = p_a + \int_z^{\eta} \rho(\zeta)\, d\zeta$$

Corollary:

$$\partial_x p(z) = \rho(\eta)\, \partial_x \eta + \int_z^{\eta} \partial_x \rho\, d\zeta$$

mass

$$\partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0$$

momentum

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_z(\rho u w)$$

$$= - \frac{1}{F^2}\left(\rho(\eta)\,\partial_x \eta + \int_z^\eta \partial_x \rho\, d\zeta\right)$$

$$+ \nu\left(\partial_{xx} u + \frac{1}{\varepsilon^2}\partial_{zz} u\right)$$

incompressibility

$$\partial_x u + \partial_z w = 0$$

Definition: f integrable in Ω, $h := \eta - b$

$$\overline{f}(x,t) := \int_{b(x)}^{\eta(x,t)} f(x,z,t)dz \quad \textbf{depth – integral}$$

$$\langle f \rangle(x,t) := \frac{1}{h(x,t)} \overline{f}(x,t) \quad \textbf{depth – average}$$

Transport Theorem: Let $f(x,z,t)$ be differentiable function. Assume kinematic boundary conditions at $z = \eta$ and $z = b$. Then

$$\int_b^\eta (\partial_t f + \partial_x(uf) + \partial_z(wf))\, dz = \partial_t \overline{f} + \partial_x \overline{uf}. \tag{1}$$

Proof:

$$\partial_t \overline{f} = (f \partial_t \eta)|_b^\eta + \int_b^\eta \partial_t f \, dz$$

$$\partial_x \overline{uf} = ((uf) \partial_x \eta)|_b^\eta + \int_b^\eta \partial_x (uf) \, dz$$

implies

$$\begin{aligned} & \partial_t \overline{f} + \partial_x \overline{(uf)} \\ = & \int_b^\eta (\partial_t f + \partial_x (uf)) \, dz + (f(\partial_t \eta + u \partial_x \eta)) \,|_b^\eta \\ = & \int_b^\eta (\partial_t f + \partial_x (uf)) \, dz + (fw)|_b^\eta \\ = & \int_b^\eta (\partial_t f + \partial_x (uf) + \partial_z (wf)) \, dz \end{aligned}$$

q.e.d.

Corollary:

$$\partial_t f + \partial_x(uf) + \partial_z(wf) = S$$

then

$$\partial_t \overline{f} + \partial_x \overline{uf} = \overline{S}.$$

Example: $f \equiv 1$, so $S = \partial_x u + \partial_z w = 0$, $\bar{f} = \bar{1} = h$.

$\partial_t h + \partial_x \overline{u} = 0$ constant density continuity equation

Example: $f = \rho$, so $S = 0$,

$\partial_t \overline{\rho} + \partial_x \overline{\rho u} = 0$ variable density continuity equation

Example: $f = q := \rho u$ (discharge), so

$$S = -\frac{1}{F^2}\partial_x p + \nu\left(\partial_{xx}u + \frac{1}{\varepsilon^2}\partial_{zz}u\right)$$

and

$$\partial_t \overline{q} + \partial_x \overline{uq} = -\frac{1}{F^2}\overline{\partial_x p} + \nu\overline{\left(\partial_{xx}u + \frac{1}{\varepsilon^2}\partial_{zz}u\right)}$$

variable density momentum equation

Assumptions:

- incompressible Navier-Stokes equations
- kinematic boundary conditions
- hydrostatic pressure

$$\partial_t \overline{\rho} + \partial_x \overline{q} = 0$$

$$\partial_t \overline{q} + \partial_x \overline{uq} = -\frac{1}{F^2}\overline{\partial_x p} + \nu \overline{\left(\partial_{xx} u + \frac{1}{\varepsilon^2}\partial_{zz} u\right)}$$

Goal: turn this into a system for $(\overline{\rho}, \overline{q})$.

Need to express

$$\overline{uq},\ \overline{\partial_x p},\ \overline{\partial_{xx} u},\ \overline{\partial_{zz} u}$$

in terms of $(\overline{\rho}, \overline{q})$

Assumption: single layer, $\rho \equiv 1$

$$\overline{\partial_x p} = \rho(\eta)\, h\, \partial_x \eta + \int_b^\eta \int_z^\eta \partial_x \rho \, d\zeta \, dz = \rho\, h\, \partial_x \eta$$

$$= \partial_x \left(\frac{1}{2} h^2 \right) + h \partial_x b$$

Assumption: constant velocity profile $u(x,z,t) \equiv u(x,t)$

$$\overline{uq} = \frac{\overline{q}^2}{h} = hu^2$$

Assumption: inviscid flow, $\nu = 0$.

Inviscid Shallow Water Equations:

$$\partial_t h + \partial_x (hu) = 0$$

$$\partial_t (hu) + \partial_x \left(hu^2 + \frac{1}{2F^2} h^2 \right) = -\frac{1}{F^2} h\, \partial_x b$$

- hyperbolic system of balance laws (for $h > 0$)
- eigenvalues

$$\lambda_\pm = u \pm \frac{\sqrt{h}}{F}$$

- non-strictly hyperbolic for $h = 0$ (dry front)

Incompressible Navier-Stokes:

– 3D domain, moving free surface
  → moving grid or level-set method

– elliptic constraint for the pressure
  → infinite propagation speeds
  → implicit solver.

Inviscid shallow water (SW):

+ 2D, fixed domain

+ hyperbolic system, finite speed of surface waves
  → explicit finite volume solver

⇒ **If SW is applicable, it is amazingly efficient!**

Chapter 2

High-Order Well-Balancing for Moving Equilibria

Balance Laws:

$$U_t + f(U)_x = s(U,x), \qquad U : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^k. \tag{2}$$

Example: 1D shallow water equations

$$U = \begin{pmatrix} h \\ m \end{pmatrix}, \; f(U) = \begin{pmatrix} m \\ \frac{m^2}{h} + \frac{g}{2}h^2 \end{pmatrix}, \; s(U,x) = \begin{pmatrix} 0 \\ -ghb_x(x) \end{pmatrix}.$$

Residuum:

$$R(U,x) \;:=\; -f(U)_x + s(U,x) \;=\; U_t \tag{3}$$

Stationary solutions (perfect balance):

$$R \equiv 0$$

Nearly stationary solutions: (near-perfect balance):

$$|R| \ll |f(U)_x| + |s(U,x)|$$

Class of balance laws with factorizable residuum:

$\exists\ c = c(U,x) \in \mathbb{R}^{k\times k}$, $V = V(U,x) \in \mathbb{R}^k$ s.t.

$$R = c\,V_x.$$

Shallow water equations: $V = (m,E)^T$, $u = m/h$,

$$E = \frac{u^2}{2} + g(h+b)$$
$$R = -(m_x, u\,m_x + h\,E_x)^T.$$

V equilibrium variables

E equilibrium energy

Examples of Equilibria

- Conservation Laws:
  Constant States
  Stationary Shocks
  Stationary Contacts
- 1D Shallow Water:
  lake at rest
  smooth river flows
  waterfalls (Noelle/Xing/Shu 2007)
- Similar Systems from Continuum Mechanics
  (Xing/Shu 2004 ff)
- 2D Shallow Water:
  geostrophic jets (coriolis force) (Bouchut/LeSommer/Zeitlin 2004, Lukacova/Noelle/Kraft 2007).
- Multi-layer Shallow Water
  (Castro/Gallardo/Pares 2006)

Semi-discrete FV:

$$\frac{d}{dt}\bar{U}_i(t) = \bar{R}_i \quad \text{on} \quad [x_{i-1/2},\, x_{i+1/2}]. \tag{4}$$

Definition: The FV scheme (4) is well-balanced for an equilibrium state $\bar{V}$ if

$$\bar{R}_i(t) = 0$$

for all data $U(t)$ such that

$$V(U(x,t),x) \equiv \bar{V}.$$

Unified treatment of 3 schemes:

• E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein and B. Perthame, *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows*, SIAM J. Sci. Comput. 25 (2004), 2050-2065.

• S. Noelle, Y. Xing and C.-W. Shu, *High order well-balanced Finite Volume WENO schemes for shallow water equation with moving water*, J. Comput. Phys. 226 (2007), 29-58.

• M. Castro, A. Pardo, C. Parés, *Well-balanced numerical schemes based on a generalized hydrostatic reconstruction technique.* Math. Mod. Meth. App. Sci. (M3AS) 17 (2007), 2055-2113.

Regular and singular parts of measures $\bar{R}_i(x)$:

$$\begin{aligned}\overline{R}_i &= \overline{R}^i_{reg} + \overline{R}^i_{sing} \\ &= \overline{R}^i_{reg} + \left(\overline{R}^{i-1/2+}_{sing} + \overline{R}^{i+1/2-}_{sing}\right)\end{aligned}$$

so

$$\frac{d}{dt}\overline{U}_i(t) = \overline{R}^i_{reg} + \overline{R}^{i-1/2+}_{sing} + \overline{R}^{i+1/2-}_{sing}$$

Theorem 1: The schemes in [ABBKP], [NXS], [CPP] satisfy

$$\overline{R}^i_{reg} = \overline{R}^{i-1/2+}_{sing} = \overline{R}^{i+1/2-}_{sing} = 0$$

for data corresponding to an appropriate equilibrium state $\overline{V}$.

Challenges for well-balancing:

regular part:

- reconstruction
- quadrature

singular part:

- simultaneous discontinuities of U and b

Smooth reconstruction in the cell interior

Hydrostatic reconstruction [ABBKP]:

$$(\bar{m}_i, \bar{\eta}_i = \bar{h}_i + \bar{b}_i, \bar{b}_i) \to (\tilde{m}, \tilde{\eta}, \tilde{b})(x).$$

Compute

$$\tilde{h}(x) := \tilde{\eta}(x) - \tilde{b}(x).$$

This preserves lake at rest.

Equilibrium reconstruction [NXS]:

Preserve all one-dimensional equilibria!

$$(\overline{U}_i, \overline{b}_i) \rightarrow (\tilde{U}, \tilde{b})(x)$$

Choose local reference values $\overline{V}_i$ by

$$\frac{1}{\triangle x_i} \int\limits_{I_i} U(\overline{V}_i, \tilde{b}(x))dx = \overline{U}_i. \tag{5}$$

Limit reconstruction according to $\overline{V}_i$.

Remainder interpolation [CPP]:

Compute $\tilde{b}(x)$, $\overline{V}_i$ as in (5).

Low order accurate equilibrium reconstruction:

$$\tilde{U}_i^*(x) := U(\overline{V}_i, \tilde{b}(x))$$

Higher order correction:

$$Q_i(x) = p(x|(\overline{U}_j - \bar{\tilde{U}}_j^*), j = i-k, \ldots, i+k)$$

Final reconstruction:

$$\tilde{U}_i(x) := \tilde{U}_i^*(x) + Q_i(x).$$

The reconstruction is well-balanced if $\overline{V}_i = \overline{V} \quad \forall i$.

Quadrature for $\overline{R}^i_{reg}$:

Given smooth reconstruction $\tilde{U}$, $\tilde{b}$

$$
\begin{aligned}
K(\tilde{R}, I_i) &\approx \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{R}(x)dx \\
&= \int_{x_{i-1/2}}^{x_{i+1/2}} (-f_2(\tilde{U})_x - g\tilde{h}\tilde{b}_x)(x)dx \\
&= -Df_2(\tilde{U}) - g \int_{x_{i-1/2}}^{x_{i+1/2}} (\tilde{h}\tilde{b}_x)(x)dx.
\end{aligned}
$$

Need to define a quadrature for the integral of the source term.

Difference calculus:

$$Da := a_R - a_L$$

$$\bar{a} := \frac{1}{2}(a_L + a_R)$$

$$D(ab) = \bar{a}Db + \bar{b}Da$$

$$\overline{(ab)} - \bar{a}\bar{b} = \frac{1}{4}DaDb$$

Lake at rest: $m \equiv 0$, $h + b \equiv \bar{\eta}$

For linear $\tilde{h}, \tilde{b}$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (\tilde{h}\tilde{b}_x)(x)dx = \frac{\tilde{h}_{i-1/2} + \tilde{h}_{i+1/2}}{2} (\tilde{b}_{i+1/2} - \tilde{b}_{i-1/2}) = \bar{h}\; D\tilde{b}$$

Therefore the quadrature

$$K(\tilde{R}, I_i) := \frac{1}{\triangle x_i} \left(-Df_2(\tilde{U}) - g\bar{h}D\tilde{b} \right)$$

is second order accurate. It is also well-balanced for lake at rest.

Moving water equilibria: $m \equiv 0,\ E \equiv \bar{E}$

$$\begin{aligned} Df_2(\tilde{U}) &= D(\tilde{m}\tilde{u} + g\tilde{h}^2/2) \\ &= \bar{m}D\tilde{u} + \bar{u}D\tilde{m} + g\bar{h}D\tilde{h} \\ &= \bar{m}D\tilde{u} + \bar{u}D\tilde{m} + \bar{h}D(\tilde{E} - g\tilde{b} - \tilde{u}^2/2) \\ &= \bar{u}D\tilde{m} + \bar{h}D\tilde{E} - g\bar{h}D\tilde{b} + (\bar{m} - \bar{h}\bar{u})D\tilde{u} \end{aligned}$$

From $\bar{m} - \bar{h}\bar{u} = D\tilde{h}D\tilde{u}/4$,

$$Df_2(\tilde{U}) = \bar{u}D\tilde{m} + \bar{h}D\tilde{E} - g\bar{h}D\tilde{b} + \frac{1}{4}D\tilde{h}(D\tilde{u})^2$$

If $D\tilde{m} = D\tilde{E} = 0$,

$$-Df_2(\tilde{U}) - g\bar{h}D\tilde{b} + \frac{1}{4}D\tilde{h}(D\tilde{u})^2 = 0$$

Well-balanced quadrature:

$$K(\tilde{R}, I_i) := \frac{1}{\triangle x_i}\left(-Df_2(\tilde{U}) - g\bar{h}D\tilde{b} + \frac{1}{4}D\tilde{h}(D\tilde{u})^2\right) \tag{6}$$

Key Difficulty: **Simultaneous jumps in U and b**

cf. nonconservative product of measures

$$-ghb_x$$

(Dal Maso/LeFloch/Murat: families of paths)

Unified framework including [ABBKP], [NXS], [CPP]

Noelle, Xing, Shu (2008). Springer-Volume on Balance Laws, ed. G. Puppo & G. Russo.

Infinitesimal layer

$$[x_{i+1/2} - \varepsilon,\ x_{i+1/2} - \varepsilon] \tag{7}$$

Continuous piecewise linear topography $\widehat{b}_\varepsilon(x)$

$$\widehat{b}_\varepsilon(x) := \begin{cases} \widetilde{b}^{\pm}_{i+1/2} & \text{for} \quad x = x_{i+1/2} \pm \varepsilon \\ \widehat{b}_{i+1/2} & \text{for} \quad x = x_{i+1/2} \pm \varepsilon/2 \end{cases}$$

Intermediate value $\widehat{b}_{i+1/2}$

$$\min\{\widetilde{b}^{-}_{i+1/2}, \widetilde{b}^{+}_{i+1/2}\} \leq \widehat{b}_{i+1/2} \leq \max\{\widetilde{b}^{-}_{i+1/2}, \widetilde{b}^{+}_{i+1/2}\}$$

Equilibrium layers in $[-\varepsilon, -\varepsilon/2]$

$$\widehat{U}_\varepsilon(x) = U(\tilde{V}_{i+1/2-}, \widehat{b}_\varepsilon(x))$$

and $[\varepsilon/2, \varepsilon]$

$$\widehat{U}_\varepsilon(x) = U(\tilde{V}_{i+1/2+}, \widehat{b}_\varepsilon(x))$$

By construction

$$\widehat{R}_\varepsilon(x) \equiv 0 \tag{8}$$

in the equilibrium layer.

Convective layer in $[-\varepsilon/2, 0] \cup [0, \varepsilon/2]$:

constant topography, so

$$\widehat{R}_\varepsilon(x) = -\partial_x \widehat{f}_\varepsilon(U(x))$$

with piecewise linear flux

$$\widehat{f}_\varepsilon(x) := \begin{cases} f(\widehat{U}_\varepsilon(x_{i+1/2} \pm \varepsilon/2)) & \text{for} \quad x = x_{i+1/2} \pm \varepsilon/2 \\ \widehat{f}_{i+1/2} & \text{for} \quad x = x_{i+1/2\pm} \end{cases}$$

and approximate homogeneous Riemann-Solver

$$\widehat{f}_{i+1/2} = f_{\mathbf{Riem}}(\widehat{U}_\varepsilon(x_{i+1/2} - \varepsilon/2), \widehat{U}_\varepsilon(x_{i+1/2} + \varepsilon/2))$$

$$\overline{R}_{sing}^{i+1/2-} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{0} \widehat{R}_{\varepsilon}(x)\, dx = -\widehat{f}_{i+1/2} + f(\widehat{U}_{i+1/2-}) \tag{9}$$

Theorem 2: The approximation (9) of the singular parts of the residuum is *well-balanced*.

Proof: Need to show that

$$\widehat{U}_{i+1/2-} = \widehat{U}_{i+1/2+}. \tag{10}$$

So suppose data are in local equilibrium, $\tilde{V}_{i+1/2-} = \tilde{V}_{i+1/2+} = \overline{V}$. Then

$$\widehat{U}_{i+1/2-} = U(\overline{V}, \widehat{b}_{i+1/2}) = \widehat{U}_{i+1/2+}, \tag{11}$$

which is (10). □

Summary: Each of the buildingblocks

regular part:

- reconstruction
- quadrature

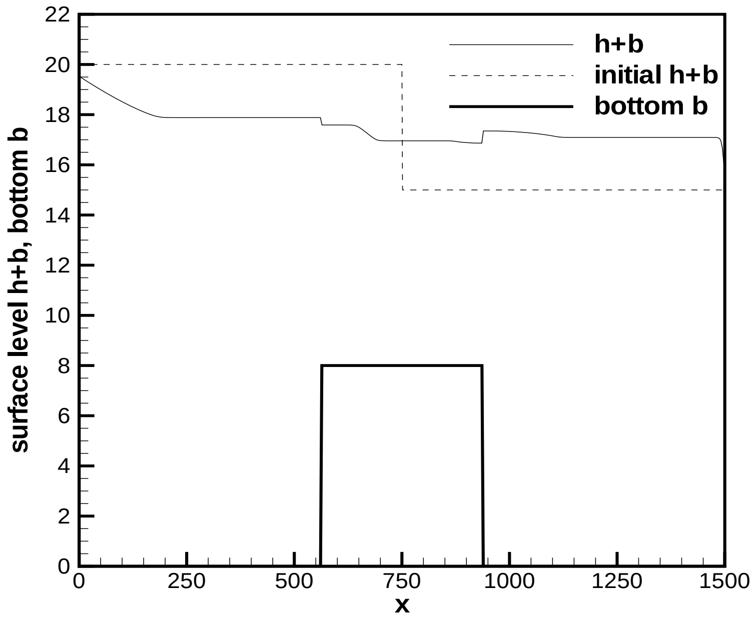
singular part:

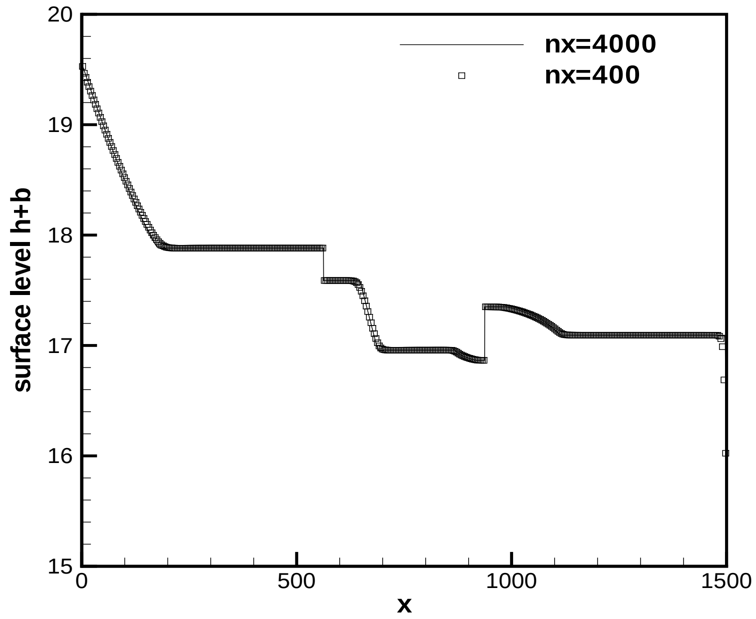
- equilibrium layer
- convective layer

is well-balanced for suitable equilibria.

This proves Theorem 1 □

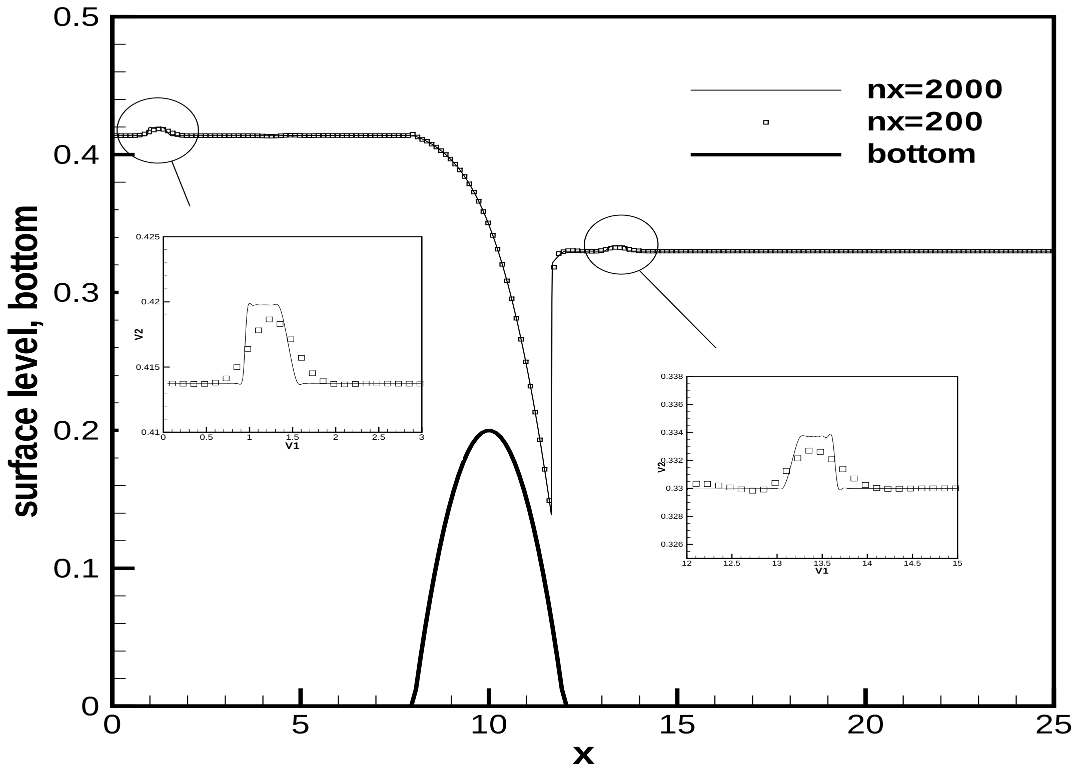
Dam break over rectangular dam





Surface level $h + b$, T=60s, 400 points.

Perturbation of 1D moving water steady state:



Small (1 %) perturbation of transcritical flow with shock

Definition: $U \in L^\infty(\Omega)$ weak solution iff $\forall \varphi \in C^1(\Omega)$

$$\iint_{\Omega} (\varphi_t U + \varphi_x f(U))\, dxdt \; - \int_{\partial\Omega} (f(U), U) \cdot n\varphi dS$$

$$= \iint_{\Omega_{reg}} \varphi\, ghb_x\, dxdt \; + \int_{\Omega_{sing}} \varphi\, g\overline{\overline{h}}Db\, dt$$

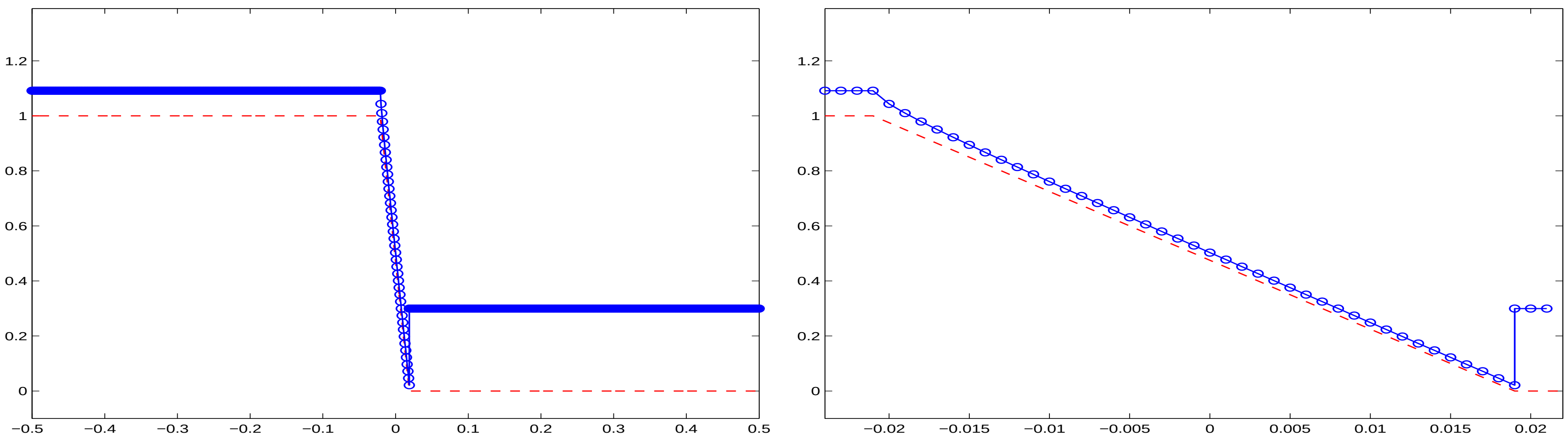
with average height in equilibrium layer

$$\overline{\overline{h}} := \frac{1}{b_R - b_L} \int_{b_L}^{b_R} h(\overline{V}, b) db$$

Theorem 3: (Noelle, Xing, Shu 2007)

Limits of the [NXS] scheme are weak solutions.

- Waterfalls [NXS 2007]



Left: surface level $h+b$ and bottom b. Right: detail

Intermediate bottom:

$$\widehat{b}_{i+1/2} = \min\{\tilde{b}^{-}_{i+1/2}, \tilde{b}^{+}_{i+1/2}\}$$

Goal:

Demonstrate the advantage of

moving-water well-balanced schemes

over

still-water well-balanced schemes

Setup:

→ Perturbations of moving-water equilibria of size ε

Algorithmic ingredients:

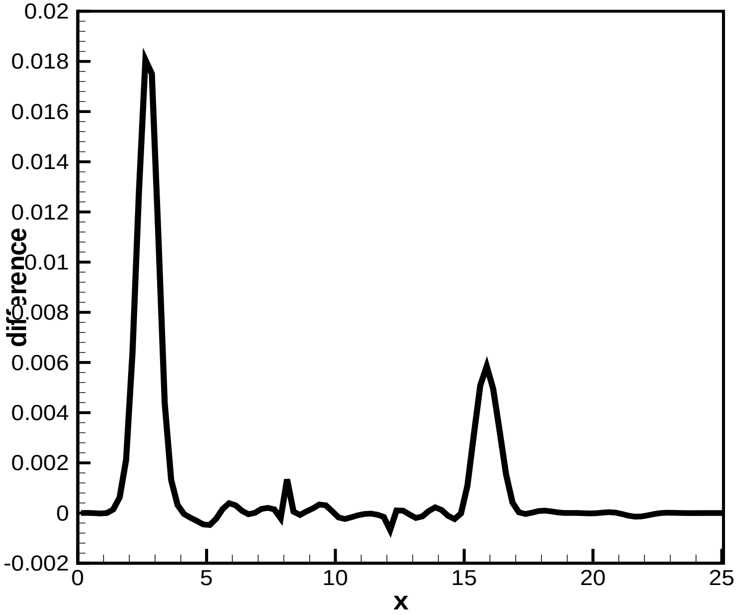
→ Shu's 5^{th} order WENO in space

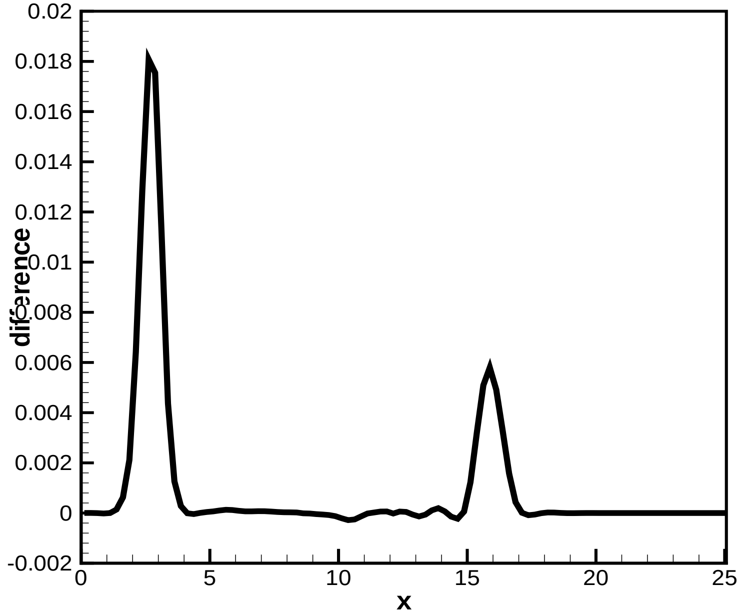
→ Shu's 3^{rd} order TVD-Runge-Kutta in time

→ Xing-Shu still-water w-b (2006)

→ Noelle-Xing-Shu moving-water w-b (2007)

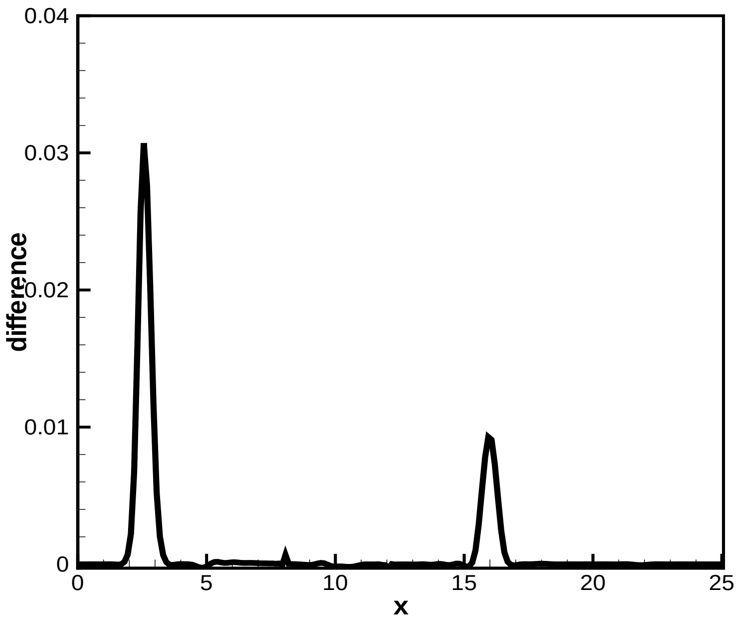
Xing, Shu, Noelle (Proceed. NumHyp2009, submitted)

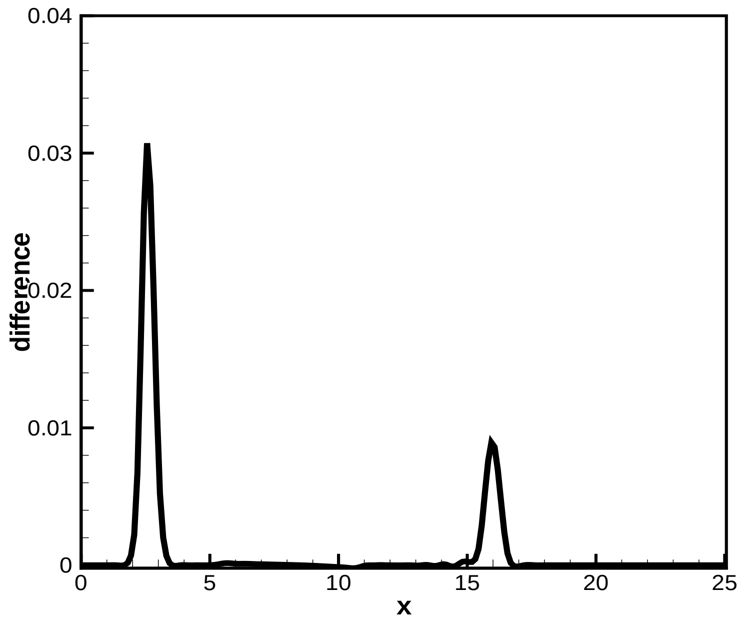




$$h - h_{equil},\ T = 1.5,\ N = 100,\ \varepsilon = 0.05.$$

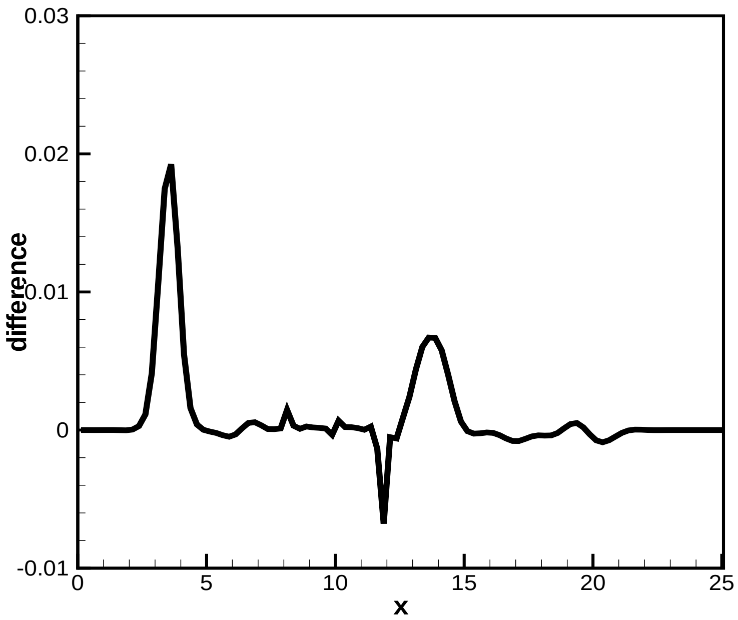
Left: still-water w-b. Right: moving-water w-b.

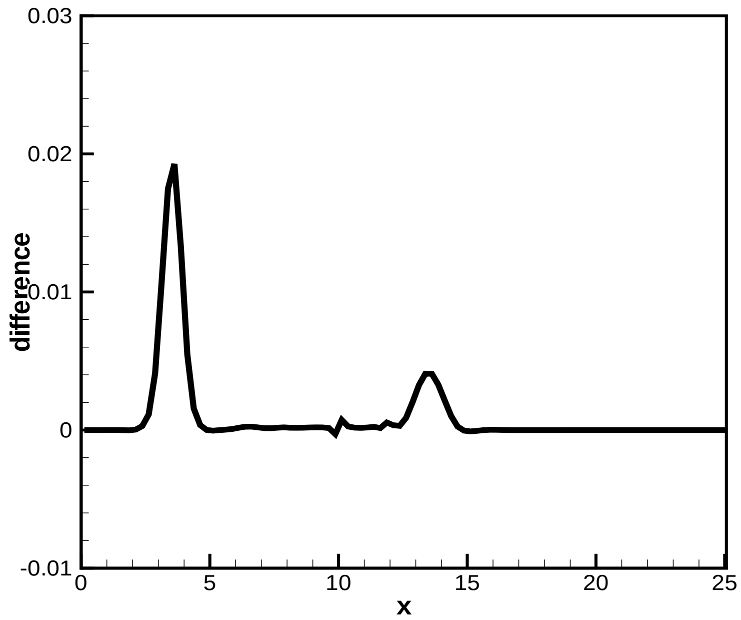




$N = 1000$, $\varepsilon = 0.05$.

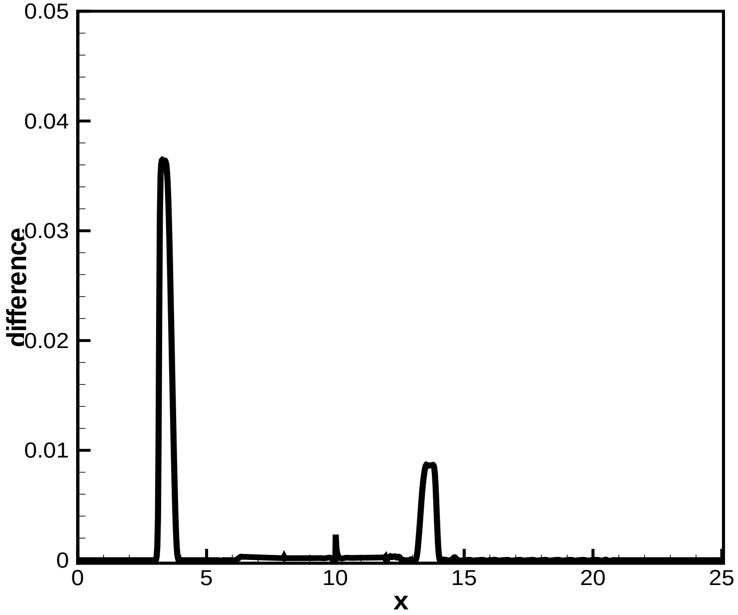
Left: still-water w-b. Right: moving-water w-b.

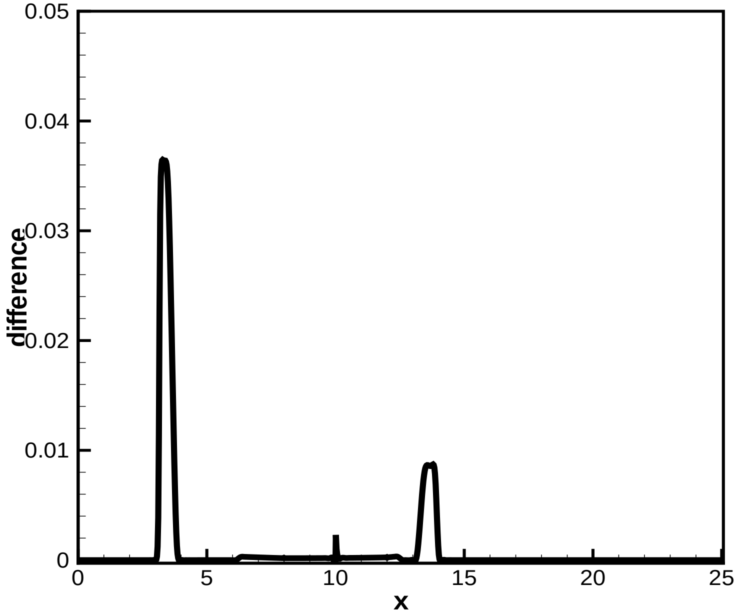




$$h - h_{equil},\ T = 1.5,\ N = 100,\ \varepsilon = 0.05.$$

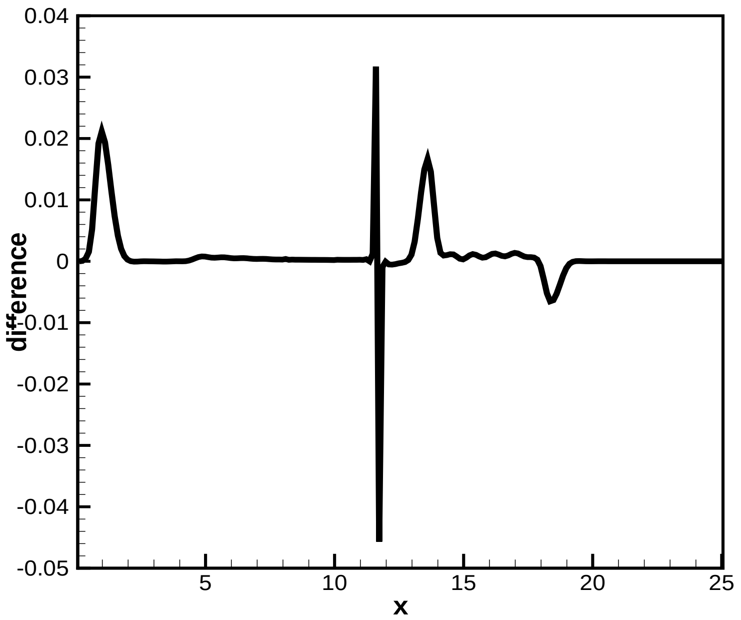
Left: still-water w-b. Right: moving-water w-b.

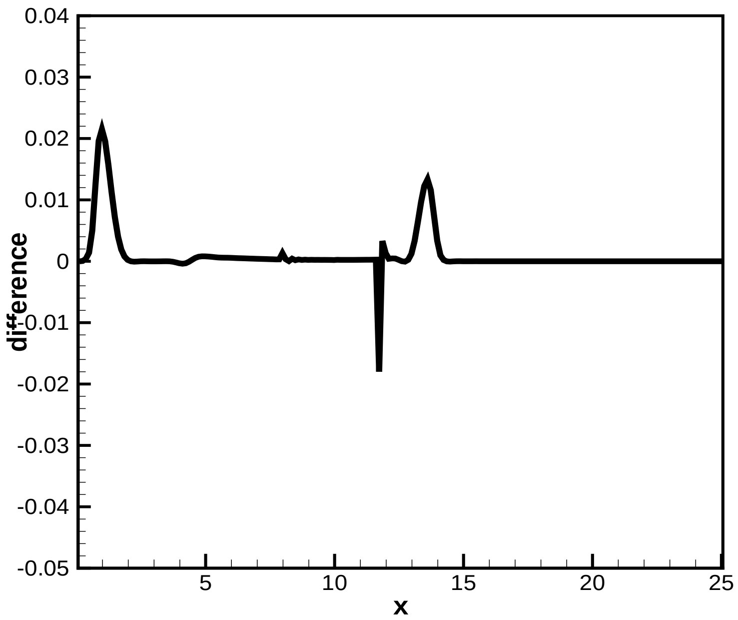




$N = 1000$, $\varepsilon = 0.05$.

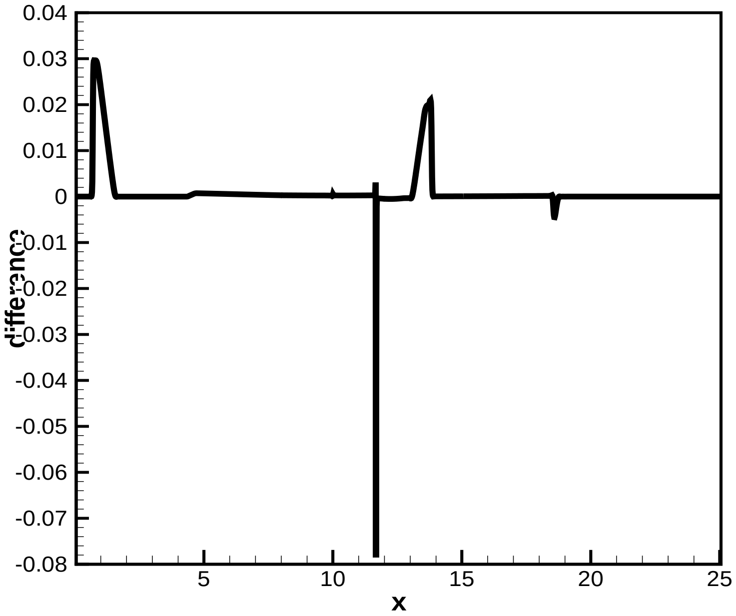
Left: still-water w-b. Right: moving-water w-b.

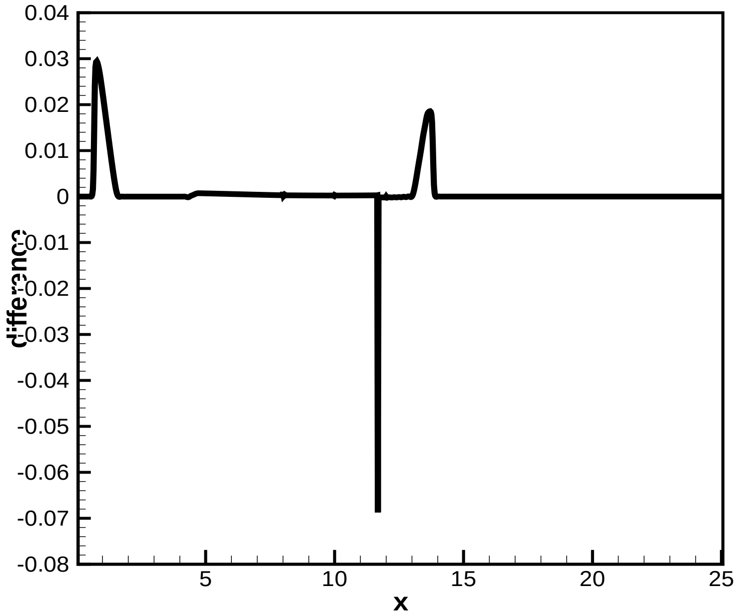




$h - h_{equil}$, $T = 3$, $N = 200$, $\varepsilon = 0.05$.

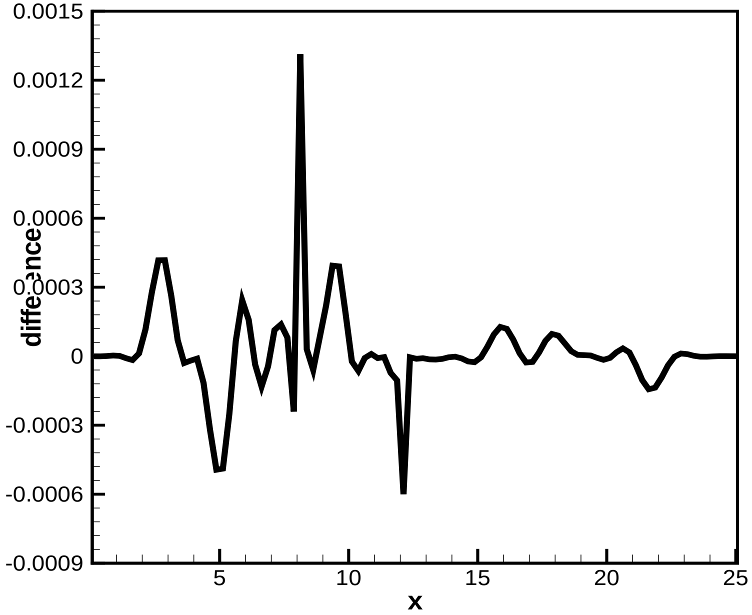
Left: still-water w-b. Right: moving-water w-b.

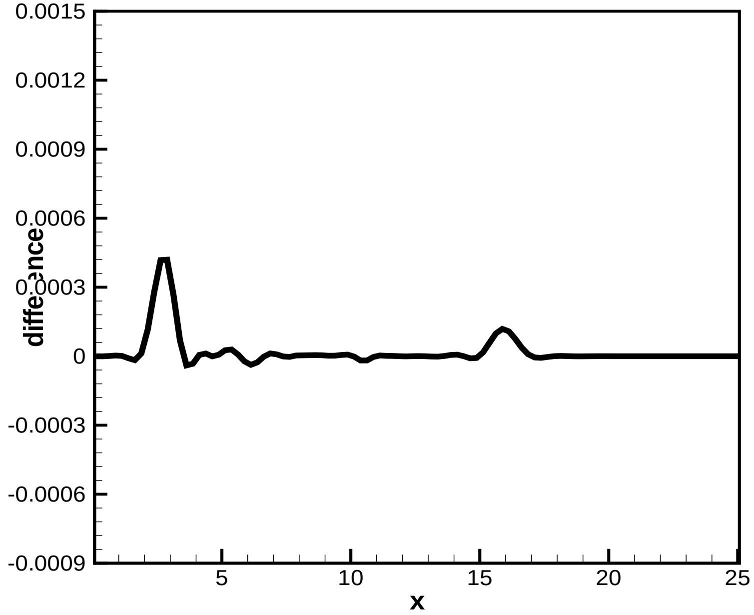




$$h - h_{equil},\ T = 3,\ N = 1000,\ \varepsilon = 0.05.$$

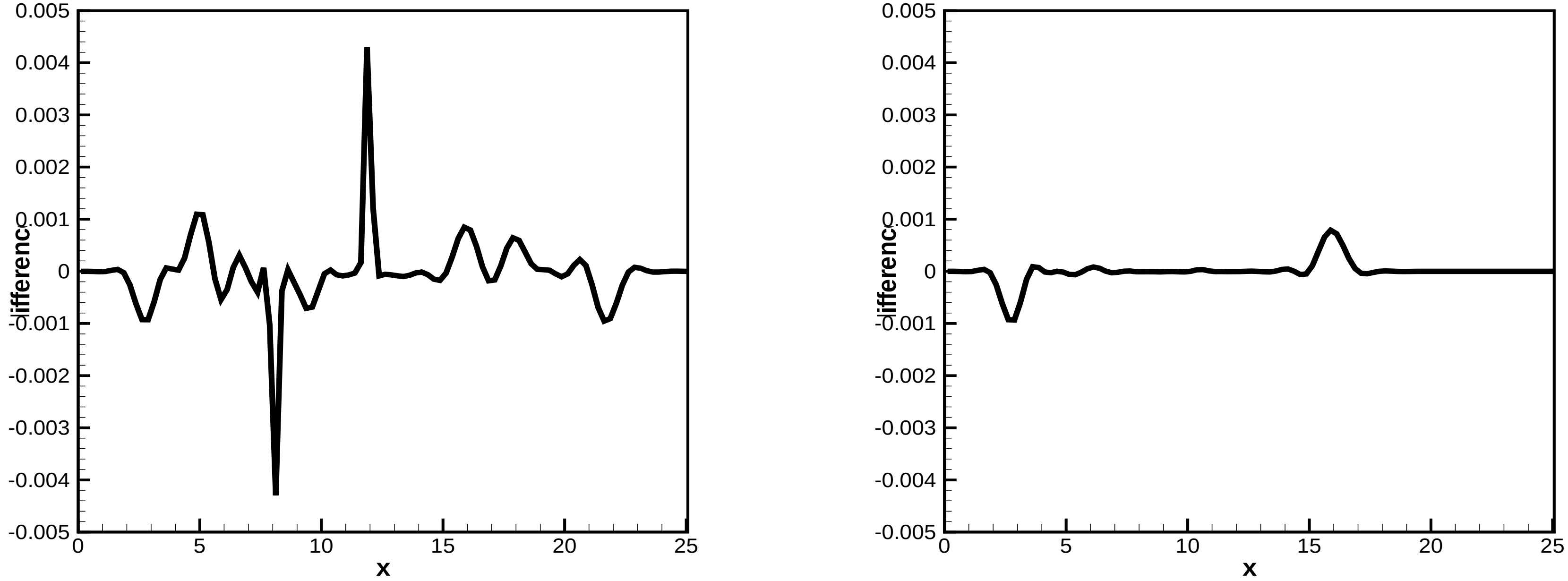
Left: still-water w-b. Right: moving-water w-b.





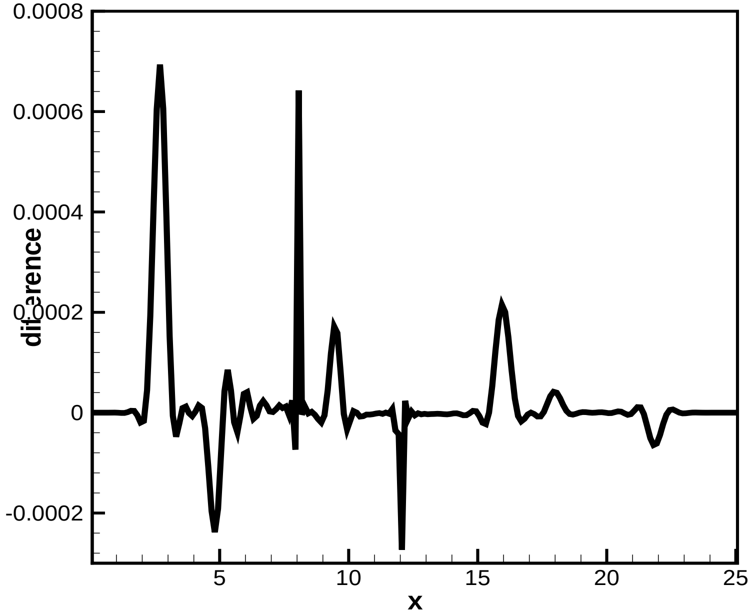
$h - h_{equil}$, $T = 1.5$, $N = 100$, $\varepsilon = 0.001$.

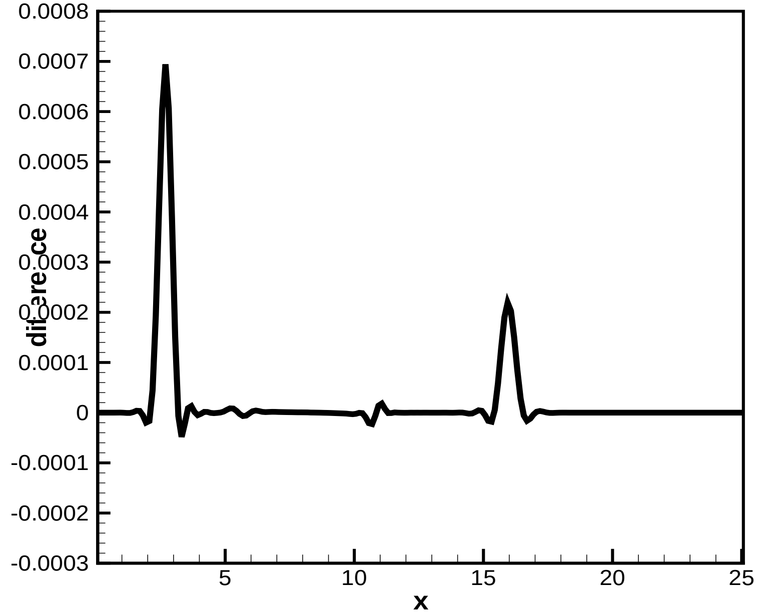
Left: still-water w-b. Right: moving-water w-b.



$$hu - (hu)_{equil},\ T = 1.5,\ N = 100,\ \varepsilon = 0.001.$$

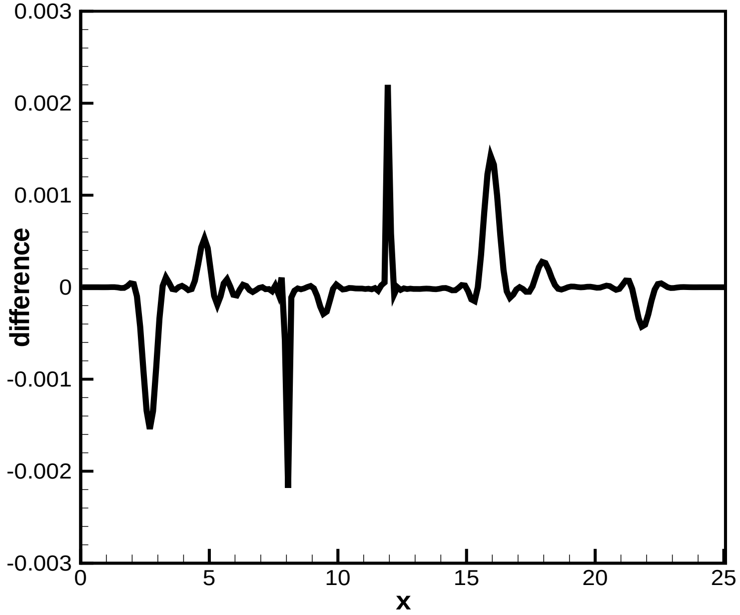
Left: still-water w-b. Right: moving-water w-b.

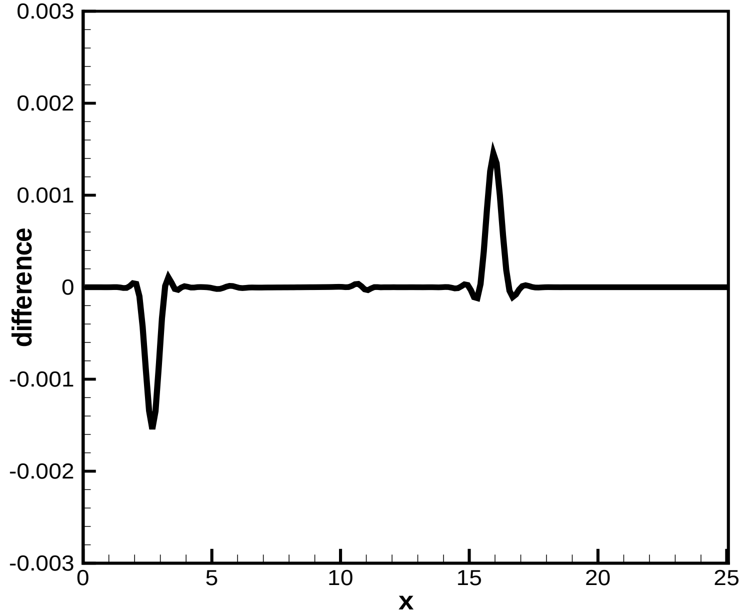




$h - h_{equil}$, $T = 1.5$, $N = 200$, $\varepsilon = 0.001$.

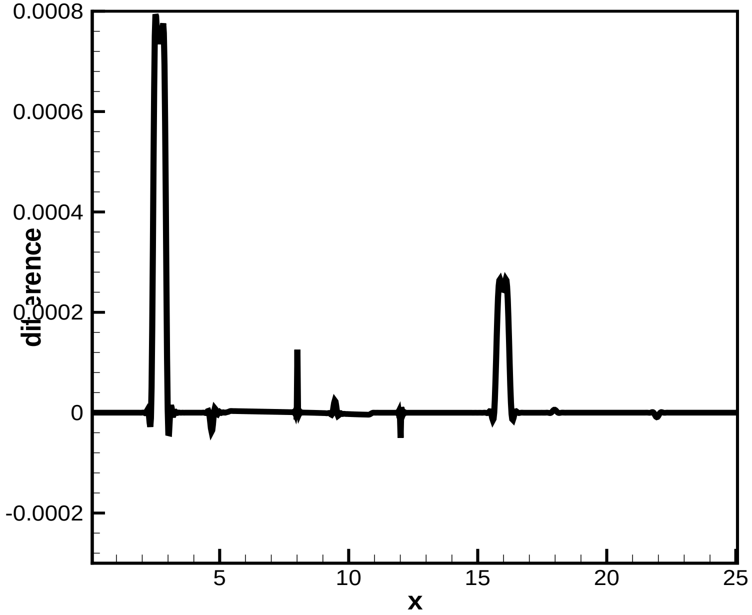
Left: still-water w-b. Right: moving-water w-b.

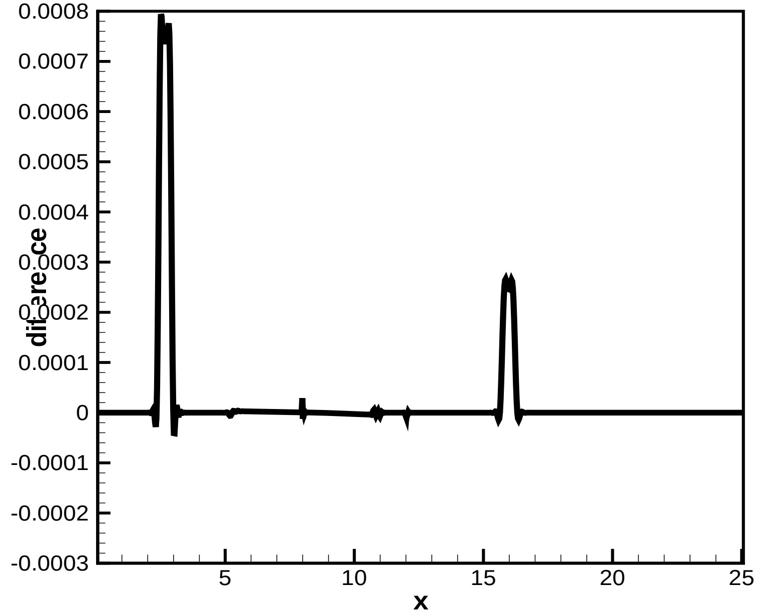




$$hu - (hu)_{equil},\ T = 1.5,\ N = 200,\ \varepsilon = 0.001.$$

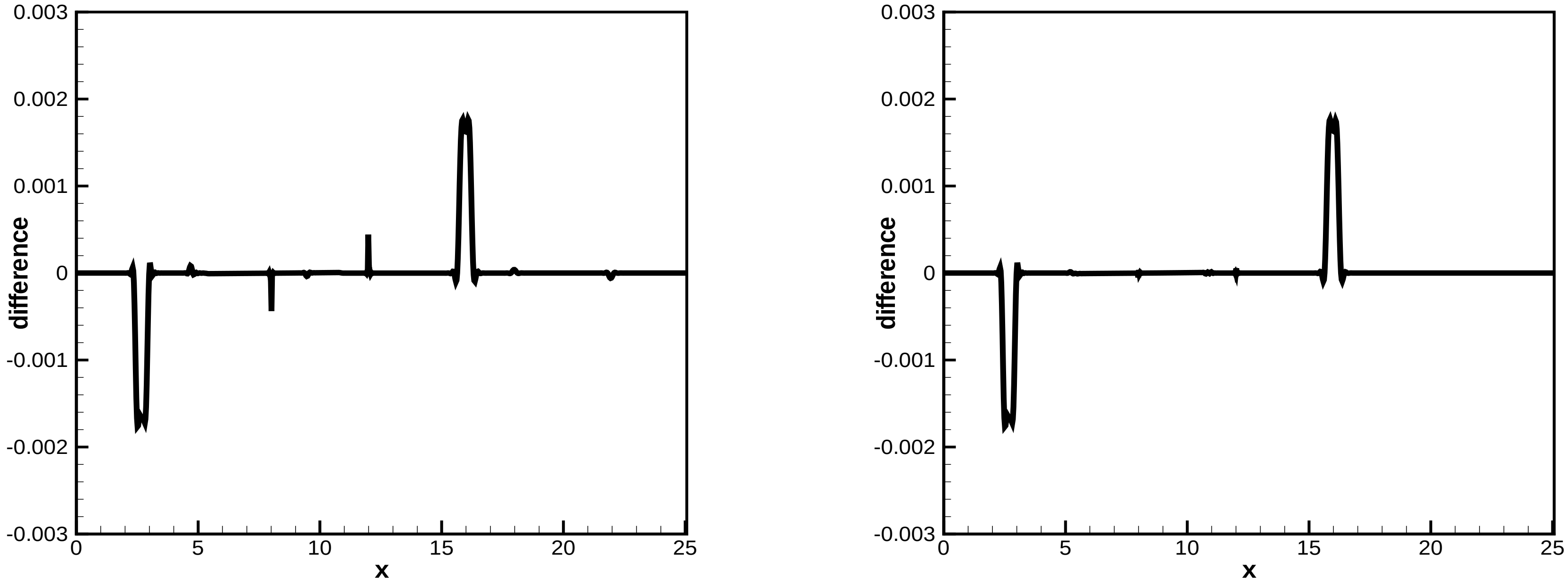
Left: still-water w-b. Right: moving-water w-b.





$$h - h_{equil},\ T = 1.5,\ N = 1000,\ \varepsilon = 0.001.$$

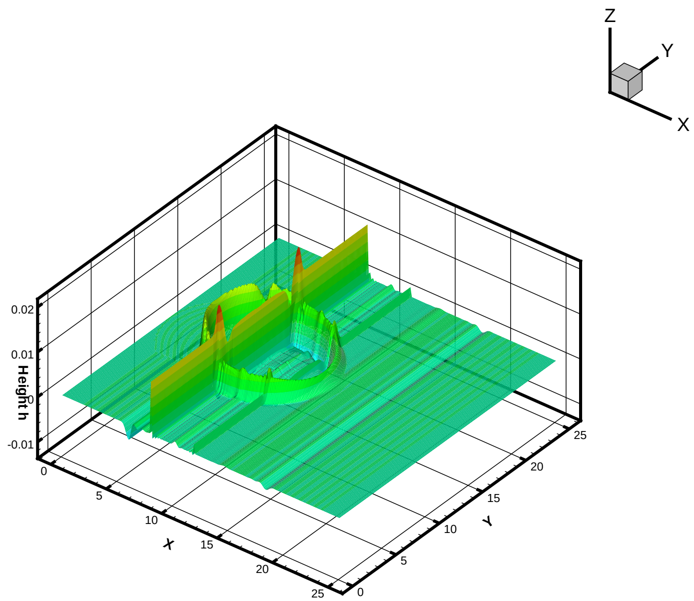
Left: still-water w-b. Right: moving-water w-b.

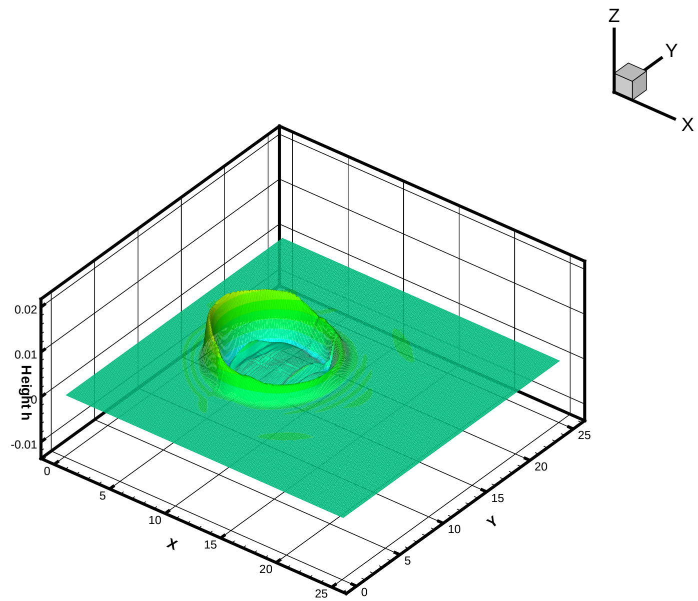


$hu - (hu)_{equil}$, $T = 1.5$, $N = 1000$, $\varepsilon = 0.001$.

Left: still-water w-b. Right: moving-water w-b.

2D perturbation of subcritical equilibrium





3D figure, $t = 1$. 200×200 points.

Left: w-b scheme for lake at rest.

Right: w-b scheme for 1D moving water.